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SOME RESULTS FOR HIGHER-ORDER VARACTOR FREQUENCY MULTIPLIERS

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TABLE OF CONTENTS

ABSTRACT

~~This report presents~~ analyses and numerical results for several frequency multipliers utilizing lossy abrupt-junction varactors are presented. The theory is based on a series model of the varactor -- a constant resistance in series with a nonlinear elastance. This model proves to be a good characterization of practical varactors and permits us to perform closed-form solutions for multipliers of any order. The problem is a nonlinear one and, consequently, it is not possible to make one solution which is applicable to all multipliers; rather, we have to solve each multiplier separately by lengthy iterative numerical calculations. Therefore, only a limited number of the infinity of possibilities are treated, ~~in this report~~.

The analysis gives formulas for general abrupt-junction-varactor frequency multipliers in the sinusoidal steady-state, i. e., we write the voltage, charge, current, and elastance as terminated Fourier-series. We derive relations for efficiency, power handling capability, input and load resistances, and bias voltage as functions of the varactor parameters and the input frequency. The results of the calculations are presented as design charts which allow one to determine the expected multiplier performance once a varactor and an input frequency are specified.

The various multipliers are compared on the basis of efficiency which results in the conclusion that they are all nearly equivalent. Chains (or cascades) of multipliers are also considered and the same conclusion results, that is, it doesn't make much difference with respect to efficiency whether lower-order multipliers are cascaded or whether a single multiplier is employed to generate a specific harmonic.

SOME RESULTS FOR HIGHER-ORDER VARACTOR FREQUENCY MULTIPLIERS

I. INTRODUCTION

In the past few years the application of the semiconductor capacitor diode (varactor) to frequency multiplication has received considerable attention. Most of this interest stems from the possibility of conversion efficiencies which are much higher than those obtainable by other techniques. Other considerations, such as power handling capability and high frequency operation, also make the varactor attractive for frequency multiplication purposes.

Several analyses of varactor frequency multipliers have been presented in recent years. These treatments have generally failed to give good results, because they are based upon either a lossless varactor model or small-signal operation of the varactor. Varactor frequency multipliers, however, are large-signal devices, and a loss mechanism -- the series resistance -- is associated with all practical varactors. Therefore, a large-signal analysis with a lossy varactor model is required for any realistic, theoretical treatment of frequency multipliers

Such an analysis has been presented by Rafuse for the abrupt-junction-varactor frequency-doubler.[1*] The techniques used by Rafuse were later extended by the author to include higher-order, abrupt-junction-varactor frequency multipliers.[2] A similar procedure has also been used in the approximate, but quite accurate, analysis of the graded-junction-varactor frequency-doubler.[3] These results (and some extensions) have now been incorporated in a book by Penfield and Rafuse.[5]

The lossy varactor model which has been used in the analyses of Rafuse,[1] Greenspan,[3] and the author[2] is shown in Fig. 1.1. In this model the lead inductance and the case capacitance have been neglected. These stray elements certainly affect the design of a practical multiplier. However,

*Superscript numerals denote references listed at the end of the report.

they do not change the fundamental limits which we seek because, in theory at least, they can be tuned out at a finite number of frequencies. Shunt conductance has also been neglected, since its impedance is large compared to that of the varactor at the operating frequencies normally used in multipliers. At low frequencies the effect of the shunt conductance must be considered. This model has been verified by careful measurements of many varactors over a wide range of frequencies and bias voltages.

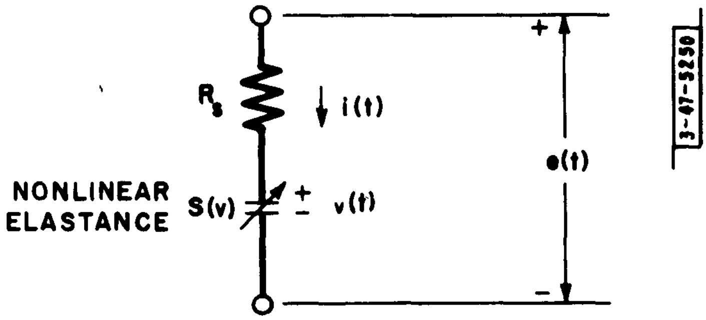

Fig. 1.1 The varactor model

The series model shown in Fig. 1.1 is most conveniently treated on an impedance basis. For this reason we consider the nonlinear element to be an elastance rather than a capacitance. This model also suggests a current- (or charge-), rather than a voltage-, controlled nonlinear characteristic, since the series resistance "gets in the way" when voltage pumping is attempted.

An idealized model of an n^{th} order frequency multiplier with idler currents is given in Fig. 1.2. The various currents flowing through the varactor are assumed to be completely separated and individually controlled in this model. In practice the only accessible currents are usually just the input and output ones (the idler circuits are often an integral part of the input and output coupling networks). The treatment of the various multipliers in this report tacitly assumes that a circuit closely approximating an idealized, lossless one can be constructed in practice (this is nearly the case for very narrow bandwidths). If the circuit losses are not small, the theoretical predictions can be corrected by very simple procedures to give a better estimate of the expected performance of the multiplier.

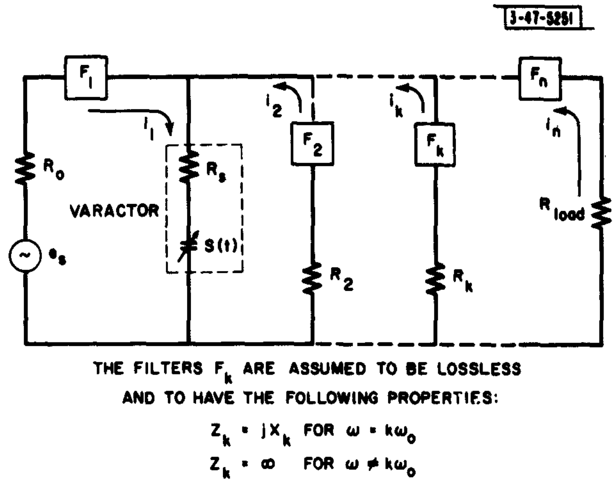

Fig. 1.2 Model for an n^{th} order multiplier with idler circuits.

In this report we present exact solutions for several higher-order, abrupt-junction-varactor frequency multipliers as follows: a tripler, two quadruplers, two quintuplers, two sextuplers, and an octupler. A tripler, a quadrupler, and a sextupler were treated previously by the author by a quasi-optimum procedure.[2] The re lts given here are true-optimum solutions. Only maximum-drive operation is considered in this report, that is, the power levels are such that the varactor is driven over its entire non-conducting (or reverse-biased) region. If necessary, the theory can easily be modified to include the under-driven case (the same numerical results are still usable).

Only a brief outline of the general multiplier theory will be given, since the derivations have been given elsewhere.[1, 2, 4, 5] The pertinent equations are given together with some discussion as to how they are obtained.

II. GENERAL MULTIPLIER ANALYSIS

2.1 Characterization of the Varactor

The varactor model which we use is shown in Fig. 1.1. It includes a constant resistance in series with a nonlinear elastance. In this report we are only concerned with the abrupt-junction varactor, which is characterized by the following incremental, nonlinear elastance:

$$S(v) = \frac{dv}{dq} = S_{max} \left[\frac{v + \varphi}{V_B + \varphi}\right]^{1/2} , \tag{2.1}$$

where φ is a constant (usually a few tenths of a volt), v is the reverse applied voltage (considered to be positive), and S_{max} is the maximum elastance which occurs at the avalanche breakdown voltage V_B. Careful measurements of many varactors have shown that Eq. (2.1) and the model of Fig. 1.1 give a quite accurate characterization of the varactor behavior. In practice, however, it is often found that a varactor has a nonzero minimum elastance, S_{min}, which occurs when $v = V_{min}$. We will use this bound rather than the one implied by Eq. (2.1), $S = 0$ when $v = -\varphi$. For this case, Eq. (2.1) is written as

$$\frac{v - V_{min}}{V_B - V_{min}} = \frac{S^2 - S^2_{min}}{S^2_{max} - S^2_{min}} . \tag{2.2}$$

Since we are dealing with a charge- rather than a voltage-controlled device, we also need the elastance-charge characteristic of the varactor. This is found by integrating Eq. (2.2):

$$S(q) = S_{max} \left[\frac{q + C}{Q_B + C}\right] , \tag{2.3}$$

where C is a constant of integration and Q_B is the charge at the breakdown voltage,

$$Q_B + C = 2S_{max} \frac{(V_B - V_{min})}{S^2_{max} - S^2_{min}} \quad . \tag{2.4}$$

The constant of integration is evaluated by requiring $S = S_{min}$ when $q = Q_{min}$ (also, $v = V_{min}$). Thus, we have

$$\frac{q - Q_{min}}{Q_B - Q_{min}} = \frac{S - S_{min}}{S_{max} - S_{min}} \tag{2.5}$$

and

$$Q_B - Q_{min} = \frac{2(V_B - V_{min})}{(S_{max} + S_{min})} \quad . \tag{2.6}$$

One final relation is required for the characterization of the varactor. It is the terminal, current-voltage relation,

$$e(t) = v(t) + R_s\, i(t) \quad , \tag{2.7}$$

or, when Eq. (2.2) is used,

$$e(t) - V_{min} = \frac{(S^2 - S^2_{min})}{(S^2_{max} - S^2_{min})} (V_B - V_{min}) + R_s\, i(t) \quad . \tag{2.8}$$

In the study of frequency multipliers, we are concerned with the fundamental frequency and the various harmonics. Therefore, we write the charge, current, voltage, and elastance in Fourier-series:

$$q(t) = \Sigma_k\, Q_k\, e^{jk\omega_o t} \quad , \tag{2.9}$$

$$i(t) = \Sigma_k\, I_k\, e^{jk\omega_o t} = \Sigma_k\, jk\omega_o\, Q_k\, e^{jk\omega_o t} \quad , \tag{2.10}$$

$$v(t) = \Sigma_k V_k e^{jk\omega_o t} , \quad (2.11)$$

$$e(t) = \Sigma_k E_k e^{jk\omega_o t} , \quad (2.12)$$

$$S(t) = \Sigma_k S_k e^{jk\omega_o t} , \quad (2.13)$$

where $Q_{-k} = Q_k^*$, $I_{-k} = I_k^*$, $V_{-k} = V_k^*$, $E_{-k} = E_k^*$, and $S_{-k} = S_k^*$, since we are dealing with real time functions. The varactor is assumed to be operated only in its non-conducting region, so there is no direct current ($I_o = 0$).

Another convenient parameter is the normalized elastance defined by

$$m(t) = \frac{S(t) - S_{min}}{S_{max} - S_{min}} = \Sigma_k M_k e^{jk\omega_o t} , \quad M_{-k} = M_k^* . \quad (2.14)$$

Since S(t) must lie between S_{min} and S_{max}, we see that

$$0 \le m(t) \le 1 . \quad (2.15)$$

For many multipliers (including most of those studied in this report) the normalized elastance-coefficients, M_k, are found to be entirely imaginary for $k \neq 0$, i.e., $jM_k = |M_k| = m_k = -jM_k^*$. Equation (2.14) for this important special case can be rewritten as

$$m(t) = m_o + 2 \Sigma_{k>0} m_k \sin k\omega_o t . \quad (2.16)$$

When the above condition is applicable, we observe that $m(t) - m_o$ is an odd function of time. Thus, m_o must equal one-half, if both limits in Eq. (2.15) are to be met. With m_o set equal to one-half, Condition (2.15) for this special case, $jM_k = m_k$ for $k > 0$, can be written as

$$\Sigma_{k>0} m_k \sin k\omega_o t \le 0.25 . \quad (2.17)$$

This completes the characterization of the abrupt-junction varactor for our purposes. Therefore, we now proceed with the formulation of the frequency multiplier characteristics (input impedance, idler and load impedances, efficiency, power handling capability, bias voltage, etc.).

2.2 Varactor Equations of Motion

An expression for the voltage at each frequency, E_k for $k \neq 0$, is found by inserting Eqs. (2.10), (2.12), (2.13), and (2.14) in Eq. (2.8):

$$E_k = R_s I_k + 2S_{min} M_k \frac{(V_B - V_{min})}{(S_{max} + S_{min})}$$
$$+ \frac{(S_{max} - S_{min})}{(S_{max} + S_{min})} (V_B - V_{min}) \Sigma_r M_r M_{k-r} \quad . \tag{2.18}$$

The current I_k can be expressed in terms of M_k by using Eqs. (2.5), (2.6), (2.9), and (2.14) in Eq. (2.10):

$$I_k = jk\omega_o Q_k$$
$$= jk\omega_o M_k (Q_B - Q_{min})$$
$$= 2jk\omega_o M_k \frac{V_B - V_{min}}{S_{max} + S_{min}} \quad . \tag{2.19}$$

When we use Eq. (2.19) in Eq. (2.18), we find

$$\frac{E_k}{I_k} = R_s + \frac{S_{min}}{jk\omega_o} + \frac{R_s \omega_c}{2jk\omega_o} \frac{\Sigma_r M_r M_{k-r}}{M_k} \quad , \tag{2.20}$$

where

$$\omega_c = \frac{S_{max} - S_{min}}{R_s} \quad , \tag{2.21}$$

is the varactor cutoff frequency as defined by Penfield.[6] (When S_{min} is negligible, ω_c reduces to the definition of cutoff frequency originally given by Uhlir.[7])

In the summation of Eq. (2. 20), the terms for $r = 0$ and $r = k$ combine with $S_{min}/jk\omega_o$ to give a term, $(S_{min} + m_o\omega_o R_s)/jk\omega_o$. It can be shown with the aid of Eq. (2. 14) that this term is the average reactance of the varactor at the k^{th} harmonic, $S_o/jk\omega_o$. Therefore, Eq. (2. 20) can be rewritten as

$$\frac{E_k}{I_k} = R_s + \frac{S_o}{jk\omega_o} + \frac{R_s\omega_c}{j2k\omega_o} \; \frac{\Sigma_{r \neq 0, k} M_r M_{k-r}}{M_k} \quad . \tag{2. 22}$$

For simplicity in the later analyses, we will assume that this reactance will be tuned out by the termination impedance at each frequency, i. e. , we choose

$$Z_k = -\frac{E_k}{I_k} = R_k + j\frac{S_o}{k\omega_o} \quad , \; k \neq \pm 1 \quad . \tag{2. 23}$$

This is not necessarily an optimum choice for the termination impedances, since improved multiplier performance may actually be obtainable in a detuned mode of operation. However, this restriction considerably simplifies the analysis and numerical evaluation of the multipliers and will, therefore, be applied in all of the following work. For this tuned mode of operation, Eq. (2. 22) becomes

$$R_s + R_k = \frac{R_s\omega_c}{2k\omega_o} \; \frac{\Sigma_{r \neq 0, k}(jM_r)(jM_{k-r})}{(jM_k)} \quad , \; k \neq \pm 1 \quad . \tag{2. 24}$$

At the input frequency we define the ratio of E_1 to I_1 as an "input impedance":

$$Z_{in} = R_s + \frac{S_o}{j\omega_o} - \frac{R_s\omega_c}{2\omega_o} \; \frac{\Sigma_{r \neq 0, 1}(jM_r)(jM_{1-r})}{jM_1} \quad . \tag{2. 25}$$

It is important to note that Z_{in} is not an impedance in the usual sense, because it depends on drive level (in particular, both S_o and the M_k change with the drive level). However, Z_{in} is a useful quantity and we shall refer to it as the "input impedance". With the same understanding, we shall call the real part of Z_{in}, Re $\left[Z_{in}\right]$, the "input resistance". It is

$$R_{in} = R_s - R_s \frac{\omega_o}{2\omega_o} \text{Re} \frac{\Sigma_{r \neq 0, 1} (jM_r)(jM_{1-r})}{jM_1} \tag{2.26}$$

An abrupt-junction-varactor frequency multiplier with idlers will have N currents (input, output, and N-2 idler currents) flowing through the varactor. These currents (or the M_k) are related by a system of (N-1) nonlinear equations as found by expanding Eq. (2.24) into its component equations. (There are actually 2(N-1) equations included in Eq. (2.24), but those for k negative are just redundant complex conjugates of those for k positive.) Note that Eq. (2.15) must always be satisfied and, in fact, it provides the N^{th} equation in the system of N nonlinear equations in the unknown M_k.

The set of equations described above give expressions for the resistances at the various harmonics in terms of the M_k. Unfortunately, the problem we must solve is usually just the inverse of this one. That is, we are given values for the idler resistances (circuit constraints) and the normalized input frequency ω_o/ω_c (varactor constraint); then we are asked to find a load resistance which will maximize either the efficiency or the power output. This requires that we "invert" the equations of motion to obtain a system of equations for the M_k in terms of the R_k and the input frequency. Since the (N-1) equations for the R_k, Eq. (2.24), are quadratic in the M_k, it is quite clear that solutions for the M_k will result in higher-order polynomials in the general case. Therefore, our problem will involve finding the roots of higher-order polynomials. This can be done in closed form only for quadratic, cubic, and quartic equations. Thus, in the general case, we will have to resort to numerical techniques in order to find the desired roots (solutions for the M_k).

A further complication in effecting a solution for a multiplier is that Condition (2. 15) contains an additional unknown, the time t_o at which m(t) is a maximum (or minimum). Thus, Condition (2. 15) in the general case is a transcendental equation which must also be solved by iterative numerical techniques.

The above considerations point out the difficulties involved in the solution of abrupt-junction-varactor frequency multipliers. Except in the simplest cases, there is obviously little hope of finding a solution for higher-order multipliers in closed form. It is clear, therefore, that we will have to resort to iterative numerical procedures in order to find the desired solutions. These calculations are long and tedious and are best done by machine computation.

A general approach to the problem of performing the calculations on a digital computer is available. However, we will defer a description of the method until we have formulated the tripler equations, because the process can best be described with reference to a particular multiplier.

Once we have found (by a suitable numerical process) the values of the M_k and the load resistance which maximize efficiency or power output, it is a simple matter to calculate the remaining parameters of the multiplier (efficiency, "input resistance", power handling capability, bias voltage, etc.). The "input resistance" equation has already been derived. Therefore, we turn now to the formulation of power relations, bias voltage formulas, and an efficiency equation.

2.3 Power Handling Capability

The input power can be computed in terms of the input current and the input resistance as follows:

$$P_{in} = 2|I_1|^2 R_{in} \ , \qquad (2.27)$$

or, with the use of Eqs. (2. 19) and (2. 21),

$$P_{in} = 8\left(\frac{S_{max} - S_{min}}{S_{max} + S_{min}}\right)^2 P_{norm} \left(\frac{\omega_o}{\omega_c}\right)^2 m_1^2 \frac{R_{in}}{R_s} , \tag{2.28}$$

where $m_1 = |M_1|$ (in general $m_k = |M_k|$) and

$$P_{norm} = \frac{(V_B - V_{min})^2}{R_s} \tag{2.29}$$

is the varactor normalization power. Alternatively, P_{in} is equal to the power dissipated in all idler resistances, in the load resistance, and in the varactor series resistance. Thus,

$$P_{in} = 8\left(\frac{S_{max} - S_{min}}{S_{max} + S_{min}}\right)^2 P_{norm}\left(\frac{\omega_o}{\omega_c}\right)^2 \sum_{\substack{k>0 \\ R_1=0}} k^2 m_k^2 \frac{(R_k + R_s)}{R_s} , \tag{2.30}$$

where in the summation for $k = 1$ we let R_1 be zero (no power dissipation in the "input resistance"). Equation (2.28) is easier to evaluate, but Eq. (2.30) shows clearly how the input power is divided up among the various frequencies

The output power is easily seen to be that portion of the input power which is dissipated in the load resistance R_n. It is given by the term in Eq. (2.30) which contains R_n:

$$P_{out} = 8\left(\frac{S_{max} - S_{min}}{S_{max} + S_{min}}\right)^2 P_{norm} \left(\frac{\omega_o}{\omega_c}\right)^2 n^2 m_n^2 \frac{R_n}{R_s} . \tag{2.31}$$

The dissipated power is simply the difference between Eq. (2.30) and Eq. (2.31). It is

$$P_{diss} = 8\left(\frac{S_{max} - S_{min}}{S_{max} + S_{min}}\right)^2 P_{norm} \left(\frac{\omega_o}{\omega_c}\right)^2 \times$$

$$\left[\sum_{k>0} k^2 m_k^2 + \sum_{\substack{k>0 \\ k \neq 1, n}} k^2 m_k^2 \frac{R_k}{R_s}\right] . \tag{2.32}$$

Of this dissipated power, a portion

$$P_{diss,v} = 8\left(\frac{S_{max} - S_{min}}{S_{max} + S_{min}}\right)^2 P_{norm}\left(\frac{\omega_o}{\omega_c}\right)^2 \Sigma_{k>0}\, k^2 m_k^2 \qquad (2.33)$$

is dissipated in the series resistance of the varactor. The remainder is the power dissipated in the idler terminations.

2.4 Efficiency

The conversion efficiency ϵ is the ratio of output power to input power. Thus, from Eqs. (2.28), (2.30), and (2.31),

$$\epsilon = \frac{n^2 m_n^2 R_n}{m_1^2 R_{in}}$$

$$= \frac{n^2 m_n^2 R_n}{\Sigma_{\substack{k>0 \\ R_1=0}}\, k^2 m_k^2 (R_k + R_s)} \;, \qquad (2.34)$$

where in the last expression we set R_1 equal to zero, since the input power is not dissipated in the "input resistance".

2.5 Bias Voltage

The bias voltage is found by taking the time average of Eq. (2.2):

$$\frac{V_o - V_{min}}{V_B - V_{min}} = \left(\frac{S_{max} - S_{min}}{S_{max} + S_{min}}\right)\left[m_o^2 + \frac{2m_o S_{min}}{S_{max} - S_{min}} + 2(m_1^2 + m_2^2 + \ldots)\right], \qquad (2.35)$$

where Eq. (2.14) has been used to write Eq. (2.35) in terms of the M_k.

An alternate expression for the bias voltage is obtained by time averaging Eq. (2.1):

$$\frac{V_o + \varphi}{V_B + \varphi} = \left(\frac{S_{max} - S_{min}}{S_{max}}\right)^2 \left[\left(m_o + \frac{S_{min}}{S_{max} - S_{min}}\right)^2 + 2(m_1^2 + m_2^2 + \dots)\right], \tag{2.36}$$

where Eq. (2.14) has again been used to write S(t) in terms of the M_k.

2.6 Idler Currents

In the preceding analysis we have not specified the number of currents in a multiplier or the frequencies at which they must flow. The simplest case, a doubler, obviously requires currents at just the fundamental and second harmonics. However, it is not so clear which currents must be present in higher-order multipliers. Therefore we must examine this important problem.

The abrupt-junction varactor has a square-law voltage-elastance (or voltage-current) characteristic, Eq. (2.2). This tells us that no multiplier, other than a doubler, can be constructed with just two currents flowing through the varactor. However, we can let extra currents flow and obtain higher-order multiplication. For example, mixing of the fundamental and the second-harmonic currents and elastances generates a voltage at the third harmonic. A tripler can therefore be constructed with an idler at the second harmonic. Doubling of the second harmonic also permits construction of a quadrupler with an idler at the second harmonic. Both of these multipliers are analyzed in this report, and they are found to give very good efficiencies as long as the operating frequency is not too high (ω_o/ω_c less than about 0.1 for efficiencies greater than 5 or 10 per cent).

Further consideration of the power transfer characteristics of the abrupt-junction varactor indicates that an idler frequency must be related in at least one of the following ways to the frequencies of the currents already present:

1. Sum of two frequencies,
2. Difference between two frequencies,
3. Twice some frequency,
4. Half some frequency.

Use of the above constraints shows that a quintupler, for example, could be constructed with idlers at $2\omega_o$ and $3\omega_o$, $2\omega_o$ and $4\omega_o$, or $2\omega_o$, $3\omega_o$, and $4\omega_o$.

This gives only the possible idler configurations for a quintupler when the idler frequencies are less than the output frequency. Obviously, there are numerous additional possibilities, if we allow idler currents to flow at frequencies higher than the output frequency.

In general, there are numerous possible idler configurations for realizing a particular higher-order multiplier (many of these are tabulated in Chapter 8 of Reference 5). Unfortunately, there is no a priori way of telling which possibility will lead to the best multiplier performance (or, indeed, whether a particular multiplier will operate efficiently). The only way to find the "best" multiplier for a specified order of multiplication is to perform detailed calculations for the various possibilities. This, of course, is a lengthy procedure which has not, as yet, been performed. In this report we treat several multipliers as follows: 1-2-3 tripler, 1-2-4 quadrupler, 1-2-3-4 quadrupler, 1-2-4-5 quintupler, 1-2-3-5 quintupler, 1-2-4-6 sextupler, 1-2-3-6 sextupler, and 1-2-4-8 octupler. This is far from an inclusive list even for the five orders of multiplication treated. However, they are representative and they include several of the configurations which are used in practical situations.

III. TRIPLER SOLUTION

We now proceed with the solution of an abrupt-junction-varactor tripler.

As previously mentioned, a tripler will not work unless idler currents are allowed to flow through the varactor. The simplest idler configuration has only one idler, at the second harmonic. However, a tripler could have two or more idlers, e.g.: $2\omega_o$ and $4\omega_o$, $2\omega_o$ and $5\omega_o$, $2\omega_o$ and $6\omega_o$, $2\omega_o$, $4\omega_o$ and $5\omega_o$, etc. The idler at the second harmonic could be eliminated by choosing idlers at $\omega_o/2$ and $3\omega_o/2$. Unfortunately, there is no a priori way of telling which of the possible idler configurations would lead to the best tripler performance. If we wanted to select the best possible tripler, we would first have to solve each one completely. This would obviously be a long and tedious process. Rather than embark on such a task, we will analyze only the 1-2-3 tripler, since our intuition suggests that it will be the simplest one to analyze and construct.

3.1 Tripler Formulas

The formulas developed in Chapter II apply to this case with all M_k zero except for $k = -3, -2, -1, 0, 1, 2,$ and 3. From Eq. (2.24) the idler and load resistance equations are

$$\frac{R_3 + R_s}{R_s} = \frac{\omega_c}{3\omega_o}\,\frac{(jM_1)(jM_2)}{jM_3}\,, \tag{3.1}$$

$$\frac{R_2 + R_s}{R_s} = \frac{\omega_c}{4\omega_o}\,\frac{(jM_1)^2 - 2(jM_1)^*(jM_3)}{jM_2}\,. \tag{3.2}$$

We now want to find the phase relationships among M_1, M_2, and M_3. To this end, let us choose the time origin such that jM_1 is real and positive and thus equal to its magnitude, m_1. Then Eq. (3.1) shows that jM_2 and jM_3 have the same phase angle, and Eq. (3.2) shows, therefore, that the phase angle of jM_2 is zero. Thus, we have the result that $jM_2 = m_2$ and $jM_3 = m_3$. In terms of the m_k, the various tripler formulas are

$$R_3 = R_s\left(\frac{\omega_c}{3\omega_o}\,\frac{m_1 m_2}{m_3} - 1\right) , \tag{3.3}$$

$$R_2 = R_s\left[\frac{\omega_c}{4\omega_o}\,\frac{m_1}{m_2}(m_1 - 2m_3) - 1\right] , \tag{3.4}$$

$$R_{in} = R_s\left[\frac{\omega_c}{\omega_o}\,\frac{m_2}{m_1}(m_1 + m_3) + 1\right] , \tag{3.5}$$

$$\frac{P_{in}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2\left[\frac{\omega_c}{\omega_o}\, m_1 m_2(m_1 + m_3) + m_1^2\right] , \tag{3.6}$$

$$\frac{P_{out}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2\left(\frac{3\omega_c}{\omega_o}\, m_1 m_2 m_3 - 9m_3^2\right) , \tag{3.7}$$

$$\frac{P_{diss}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2\left[\frac{\omega_c}{\omega_o}\, m_1 m_2(m_1 - 2m_3) + m_1^2 + 9m_3^2\right] , \tag{3.8}$$

$$\frac{P_{diss,v}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2(m_1^2 + 4m_2^2 + 9m_3^2) . \tag{3.9}$$

$$\epsilon = \frac{3m_3}{m_1}\,\frac{\frac{\omega_c}{\omega_o}\, m_1 m_2 - 3m_3}{\frac{\omega_c}{\omega_o}\, m_2(m_1 + m_3) + m_1} , \tag{3.10}$$

$$\frac{V_o + \varphi}{V_B + \varphi} = \left(\frac{S_{max} - S_{min}}{S_{max}}\right)^2\left[\frac{1}{4} + 2(m_1^2 + m_2^2 + m_3^3)\right] + \frac{S_{min}}{S_{max}} , \tag{3.11}$$

where we have set $m_o = \frac{1}{2}$. For convenience, we have defined another normalization power

$$P'_{norm} = \left(\frac{S_{max} - S_{min}}{S_{max} + S_{min}}\right)^2 P_{norm} \quad . \tag{3.12}$$

The M_k have been shown to be entirely imaginary. Thus, Condition (2.17) can be used in place of Condition (2.15). The limit on the magnitudes of the m_k for the tripler, therefore, becomes

$$m_1 \sin \omega_o t + m_2 \sin 2\omega_o t + m_3 \sin 3\omega_o t \leq 0.25 \tag{3.13}$$

for all values of t.

3.2 Solution of the Tripler Equations

The above formulas are all written in terms of varactor parameters (R_s, ω_c, P'_{norm}) and the m_k. Hence, if we know values of m_1, m_2, and m_3 which satisfy Condition (3.13) and which give positive values for R_2 and R_3, it is apparent that a few simple calculations will yield the remaining multiplier parameters.

More often, as pointed out in Section 2.2, we are asked to determine the values of R_2 and R_3 which will maximize either the efficiency or the power handling capability. For every value of input power within the bound of Condition (3.13), there will be optimum values for R_2 and R_3. Usually, however, we wish to utilize the full power handling capability of the varactor; that is, we want the elastance to attain the values of S_{min} and S_{max} during each cycle. Therefore, we seek solutions for the m_k such that Condition (3.13) is satisfied with the equality sign at the time, t_o, at which m(t) is a maximum. For this maximum drive level operation, the solutions should be such that the efficiency or power output is maximized.

We are dealing with a nonlinear problem so we expect some difficulty in finding the optimum values of R_2 and R_3 for a specific input frequency, ω_o/ω_c. Part of the difficulty resides in having too many unknowns and not enough equations, i.e., we have six unknowns, m_1, m_2, m_3, R_2, R_3, and t_o, and four equations. (3.3), (3.4), (3.13), and the derivative of (3.13) evaluated at t_o

$$\left.\frac{dm(t)}{d\omega_o t}\right|_{t=t_o} = m_1 \cos \omega_o t_o + 2m_2 \cos 2\omega_o t_o + 3m_3 \cos 3\omega_o t_o = 0. \quad (3.14)$$

We can eliminate R_2 as an unknown, since its value is usually set by the practical circuit in which the varactor is imbedded (in our calculations we allow R_2 to take on the fixed values 0, R_s, $2R_s$, $5R_s$, and $100R_s$). In theory we could now solve Eqs. (3.3), (3.4), (3.13), and (3.14) for m_1, m_2, m_3, and t_o in terms of R_3. These solutions could then be used to write the efficiency in terms of R_3. Next, the optimum load resistance would be determined by equating to zero the derivative of the efficiency with respect to R_3. This is a mathematically rigorous approach, but it meets with considerable practical difficulty because Eq. (3.14) has two or more roots in the interval, $0 \leq \omega_o t \leq 2\pi$, and we must, of course, select the correct one. In higher-order multipliers the problem becomes considerably more complicated, so we will abandon this approach and turn to an iterative numerical procedure which has general applicability.

To facilitate our calculations, we solve Eqs. (3.3) and (3.4) for m_2 and m_3 in terms of m_1:

$$\frac{m_2}{m_1} = \frac{1}{\frac{R_2 + R_s}{R_s} \frac{4\omega_o}{m_1 \omega_c} + \frac{2m_1\omega_c}{3\omega_o} \frac{R_s}{R_3 + R_s}} \quad (3.15)$$

and

$$\frac{m_3}{m_1} = \frac{1}{2 + \frac{R_2 + R_s}{R_s} \frac{R_3 + R_s}{R_s} \frac{12\omega_o^2}{m_1^2 \omega_c^2}} . \quad (3.16)$$

The value of R_2 will be fixed for reasons discussed above. It is also logical to assign a value to the normalized input frequency, ω_o/ω_c, since we are usually interested in using a particular varactor for a specific application.

These parameters can, of course, be changed at a later stage of the calculations to find the expected multiplier behavior with different operating conditions.

We have no way of fixing the optimum load resistance, so we will determine the correct value by a trial and error procedure. That is, we will assign an initial value to R_3 and compute the multiplier performance; then R_3 will be varied and the calculations repeated. This process will be continued until the optimum load resistance has been located. Obviously, we can use the results of the first two calculations of R_3 to decide on the third value, etc. Finally, when we have values for R_3 which give us efficiencies (or power outputs) near the maximum, we can use parabolic interpolation through three points to get an accurate value for the optimum load resistance.

The problem has now been reduced to the solution of two equations, (3.13) and (3.14), in two unknowns, m_1 and t_o (m_2 and m_3 as given by Eqs. (3.15) and (3.16) depend only on m_1 when we assign values to ω_o/ω_c, R_2, and R_3). This pair of equations is still extremely difficult to solve, so we adopt an iterative numerical approach. To do this, we assign an initial value to m_1 which we label $m_1(1)$, then we compute $m_2/m_1(1)$ and $m_3/m_1(1)$. Next, we search for the value of t_o which satisfies Eq. (3.14). (There will be two or more solutions for t_o in the interval, $0 \le t_o \le 2\pi/\omega_o$, each of which must be investigated.) Finally, we check Condition (3.13) to see whether our initial choice of m_1 is correct at the time t_o where $m(t)$ is a maximum. If $m_1(1)$ has the wrong value, we must assign a new value to m_1 and recompute m_2/m_1, m_3/m_1, and t_o. This process is continued until we have finally located the correct value for m_1.

At this point we obviously need a systematic method of choosing successive values for m_1 such that our solution will eventually converge. Condition (3.13) evaluated at $t = t_o$ provides the necessary tool. For our purposes, we rewrite Condition (3.13) as follows

$$m_1 \left[\sin \omega_o t_o + \frac{m_2}{m_1} \sin 2\omega_o t_o + \frac{m_3}{m_1} \sin 3\omega_o t_o \right] \le 0.25 \quad . \qquad (3.17)$$

The solutions which we seek are for maximum drive level in which case we insist that Eq. (3.17) be satisfied with the equality sign. This can only occur for one value of m_1. The m_1 which we have factored out in Eq. (3.17) is not independently specifiable, but we can at least obtain an estimate of m_1 for our second iteration by assigning to it a value such that the equality in Eq. (3.17) will be met. More generally, the $(\ell + 1)^{st}$ iterative value, $m_1(\ell + 1)$, can be obtained from the ℓ^{th}, $m_1(\ell)$, by requiring

$$m_1(\ell+1) = \frac{1}{4}\left[\sin \omega_o t_o^{(\ell)} + \frac{m_2(\ell)}{m_1(\ell)} \sin 2\omega_o t_o^{(\ell)} + \frac{m_3(\ell)}{m_1(\ell)} \sin 3\omega_o t_o^{(\ell)}\right]^{-1} . \tag{3.18}$$

The numerical procedure should now be clear. We simply choose an initial value for m_1, find m_2/m_1, m_3/m_1, and t_o, and then compute a new value for m_1 from Eq. (3.18). When m_1 is changed, it is obvious that m_2, m_3, and t_o will change so the calculations must be repeated. We then find a third value for m_1 from Eq. (3.18). The process thus continues until $m_1(\ell + 1)$ and $m_1(\ell)$ agree to sufficient accuracy (the results presented in this report were calculated to an accuracy of 0.01 per cent).

Once the above calculations have been performed, it is a simple matter to use the results to find the remaining multiplier parameters (R_{in}, P_{in}, P_{out}, P_{diss}, etc.) for the given values of idler resistance and input frequency. A typical plot of efficiency, power input, and power output versus load resistance is given in Fig. 3.1. In this figure we note that the three maxima occur at slightly different values of R_3. However, the efficiency is not significantly reduced where the power output is maximized, and vice versa. For practical purposes, therefore, power output and efficiency can be simultaneously maximized.

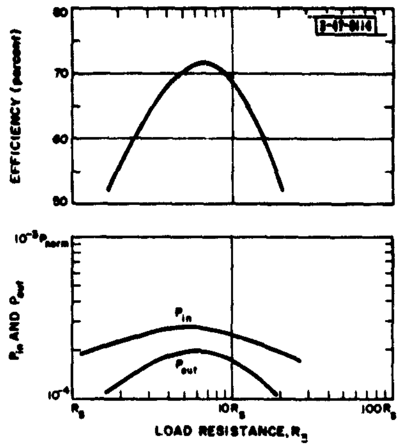

Fig. 3.1 Efficiency, power input, and power output as functions of the load resistance for an abrupt-junction-varactor tripler. In this plot we have assumed that $R_2 = 0$ and $\omega_o = 10^{-2}\omega_c$, but the qualitative features are the same for other idler resistances and frequencies.

Power output and efficiency both have very broad maxima as functions of R_3. This feature is very convenient in practical designs, since it is not always a simple matter to obtain an exact resistance for the load. Interstage matching networks in multiplier chains are also simplified in some cases because of this property. The reasonably good predictions of previous quasi-optimum solutions were also a consequence of these broad maxima.[2]

The above procedures give a value for the optimum load resistance for fixed values of idler resistance and input frequency. To generate a complete set of multiplier performance data, we vary R_2 or ω_o/ω_c (or both) and repeat the calculations. These long and tedious computations were programmed for numerical evaluation on an I. B. M. 7090 digital computer. The results for maximum efficiency operation of the tripler are given in Figs. 3.2 to 3.8. Efficiency, input resistance, load resistance, power input, power output, dissipated power, and bias voltage are plotted as functions of frequency for several values of idler resistance, R_2.

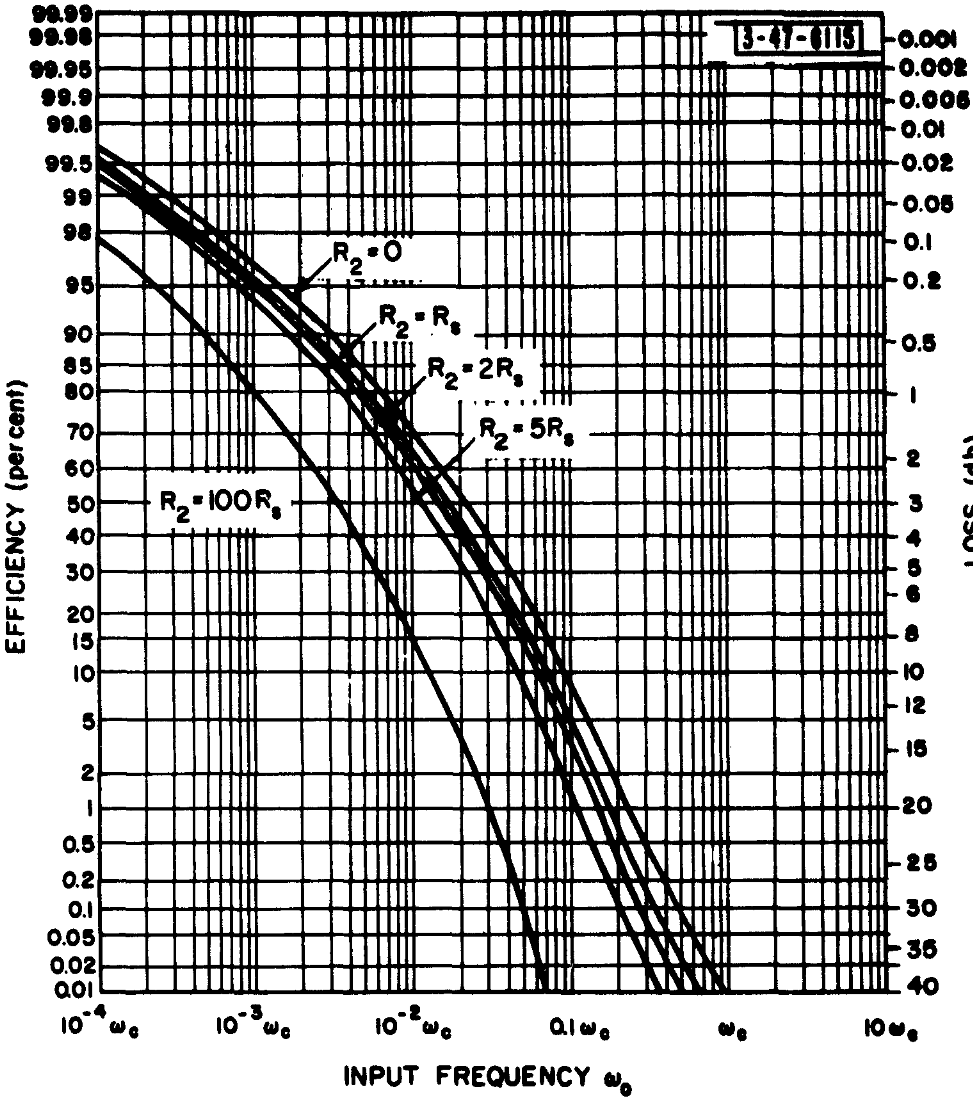

Fig. 3.2 Maximum efficiency of a 1-2-3 abrupt-junction-varactor tripler for various idler resistances, R_2. It is assumed that the varactor is fully driven, and that the load is tuned and adjusted to give maximum efficiency.

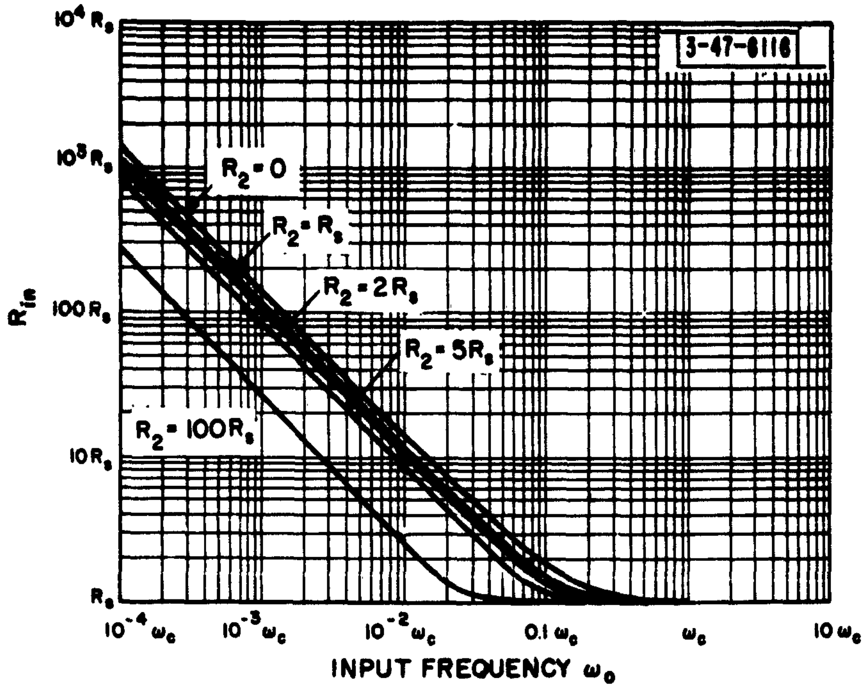

Fig. 3.3 Input resistance of an abrupt-junction-varactor tripler, adjusted to give maximum efficiency for the various values of idler resistance, R_2.

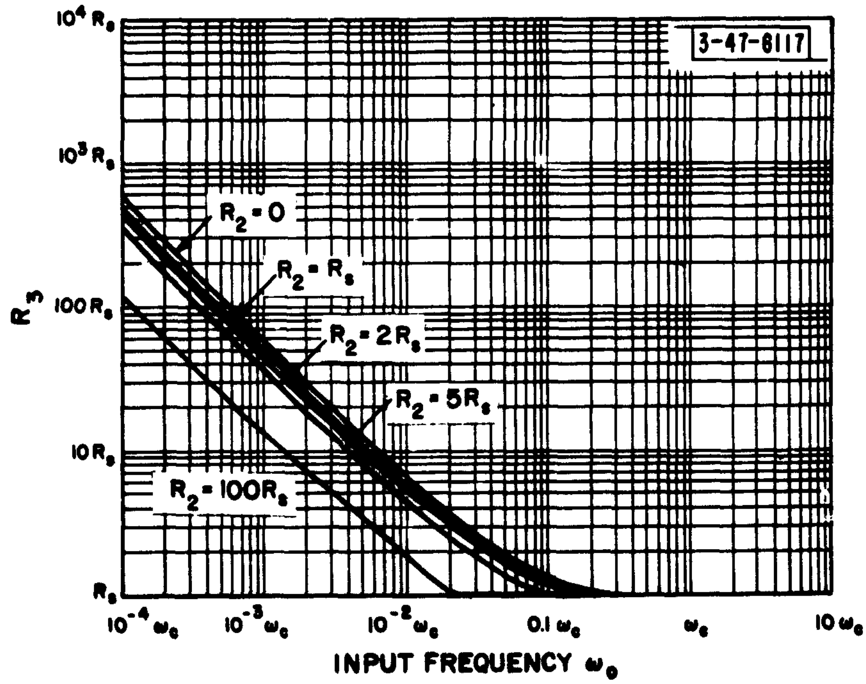

Fig. 3.4 Load resistance vs. input frequency for maximum efficiency operation of a 1-2-3 abrupt-junction-varactor tripler.

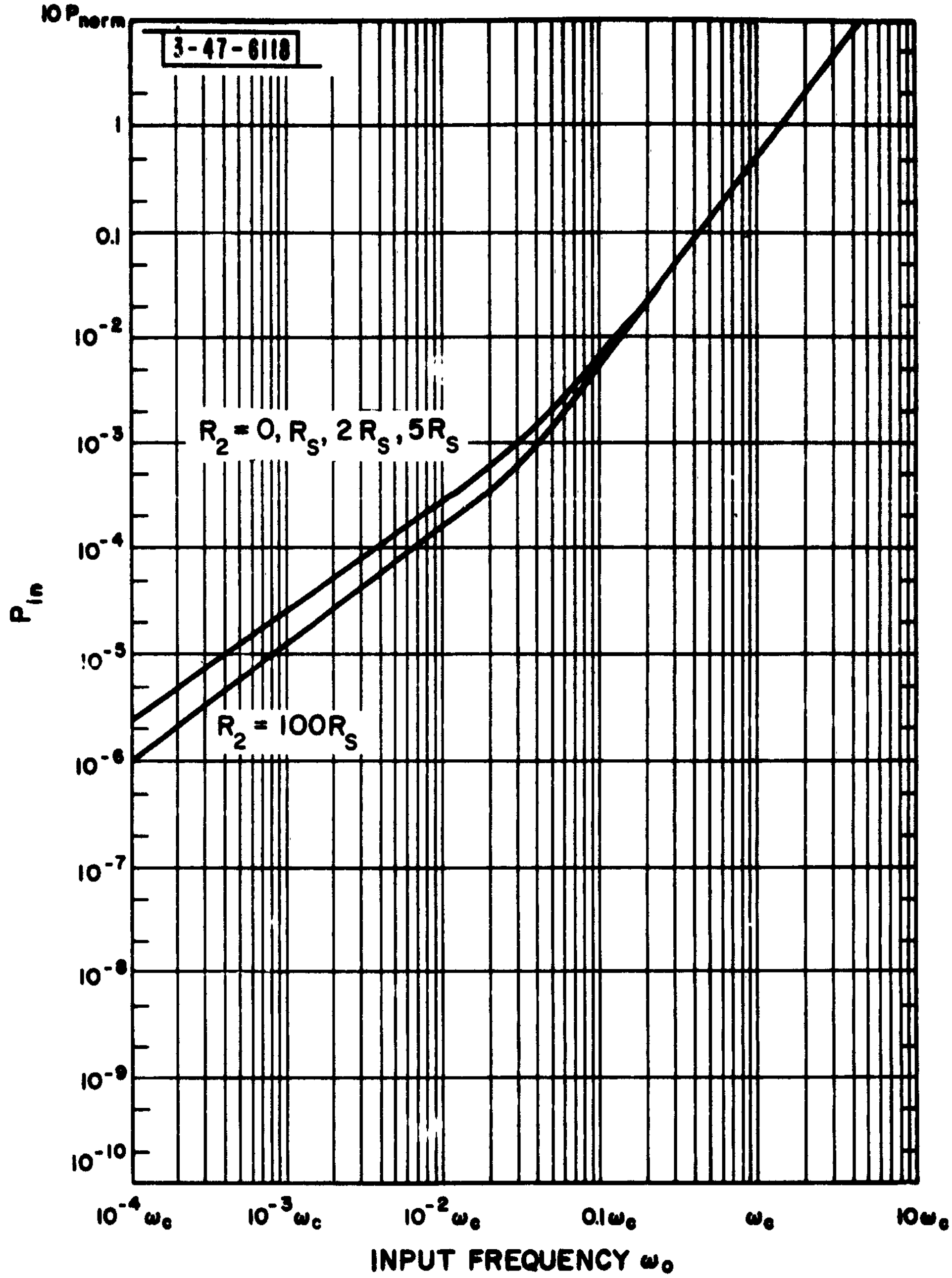

Fig. 3. 5 Input power of an abrupt-junction-varactor tripler adjusted for maximum efficiency operation for various values of idler resistance, R_2.

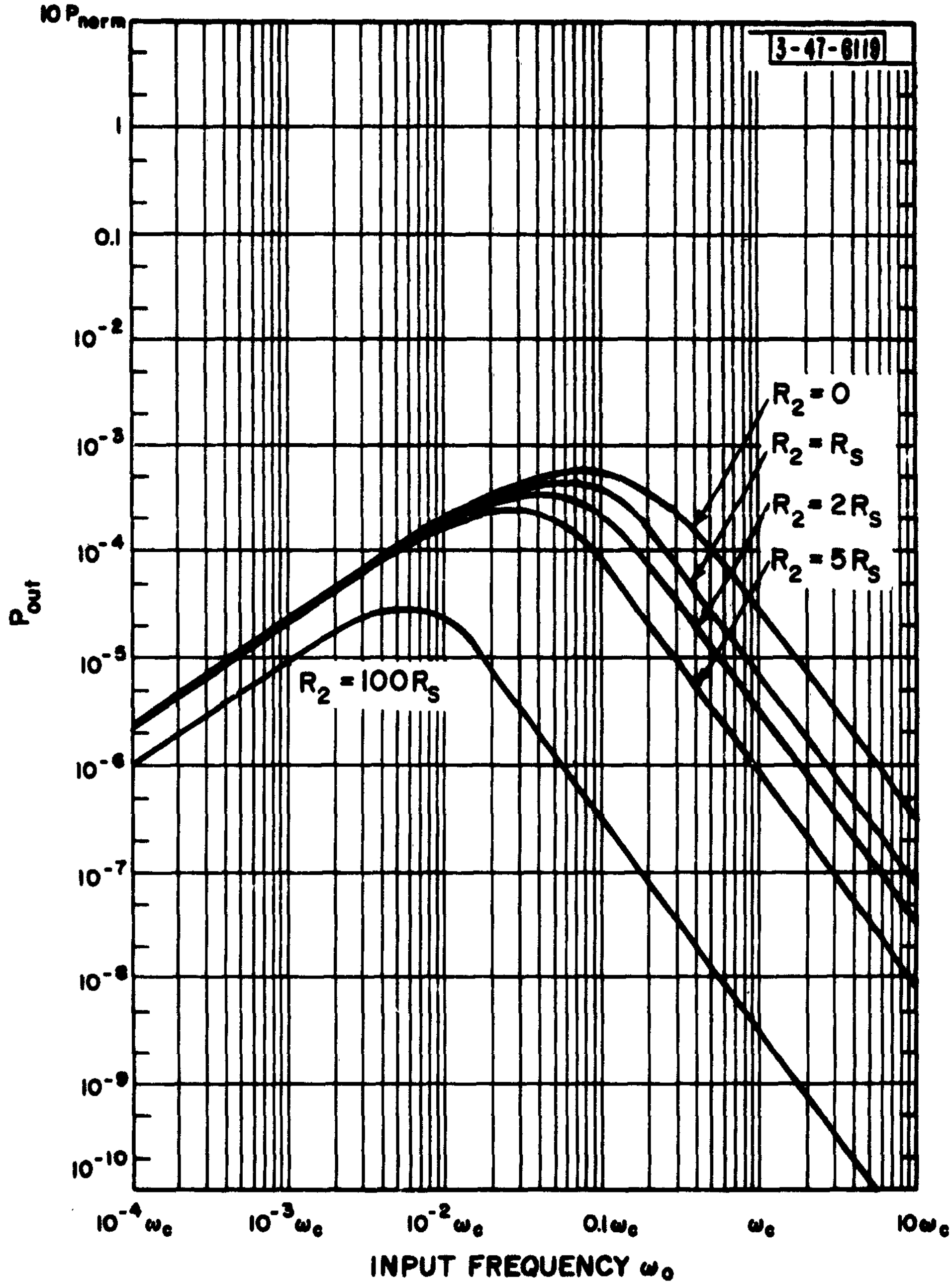

Fig. 3.6 Output power of a 1-2-3 abrupt-junction-varactor tripler for maximum efficiency operation.

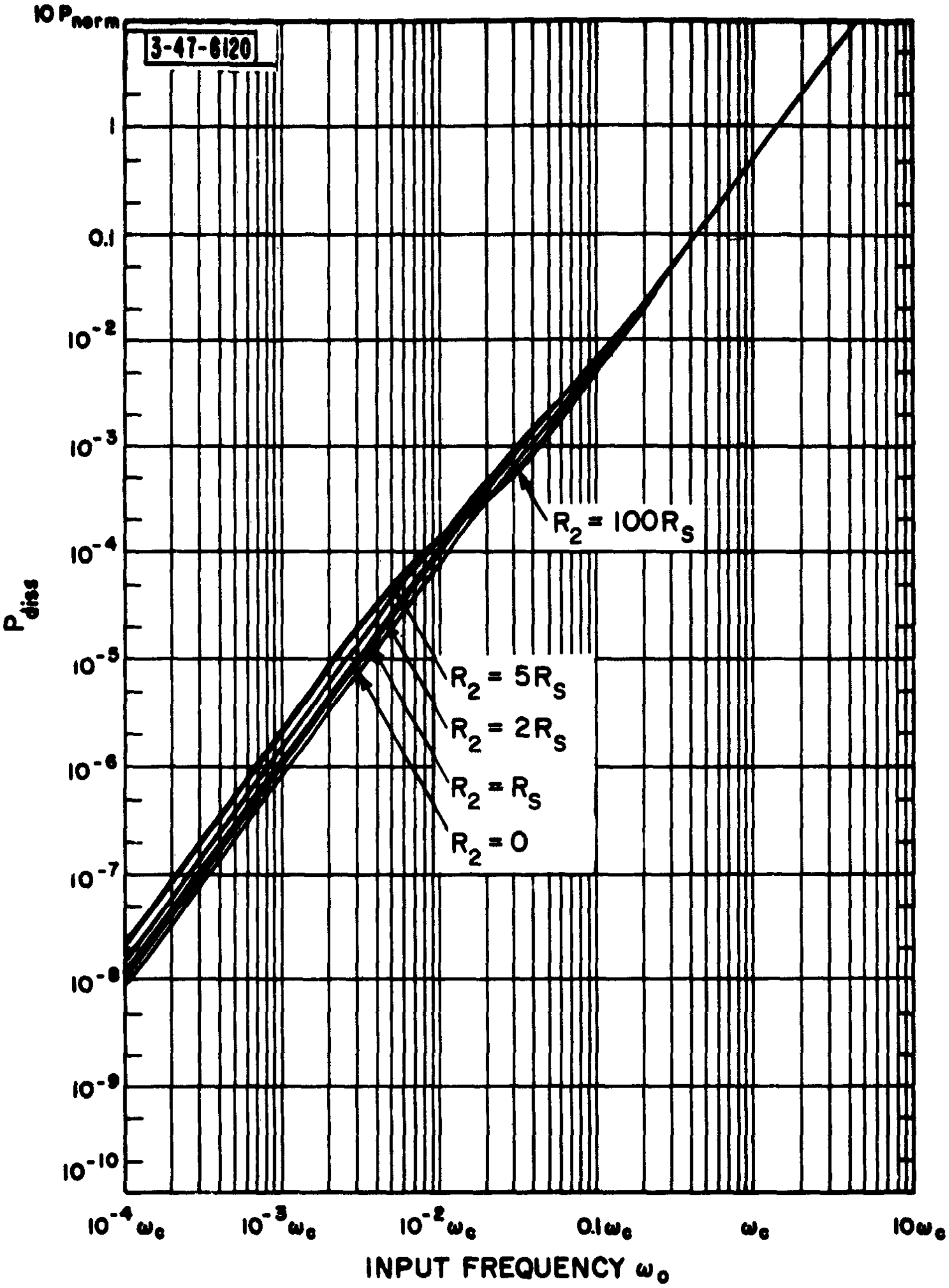

Fig. 3.7 Total dissipated power (power dissipated in the varactor plus the power dissipated in the idler termination) for an abrupt-junction-varactor tripler adjusted for maximum efficiency operation.

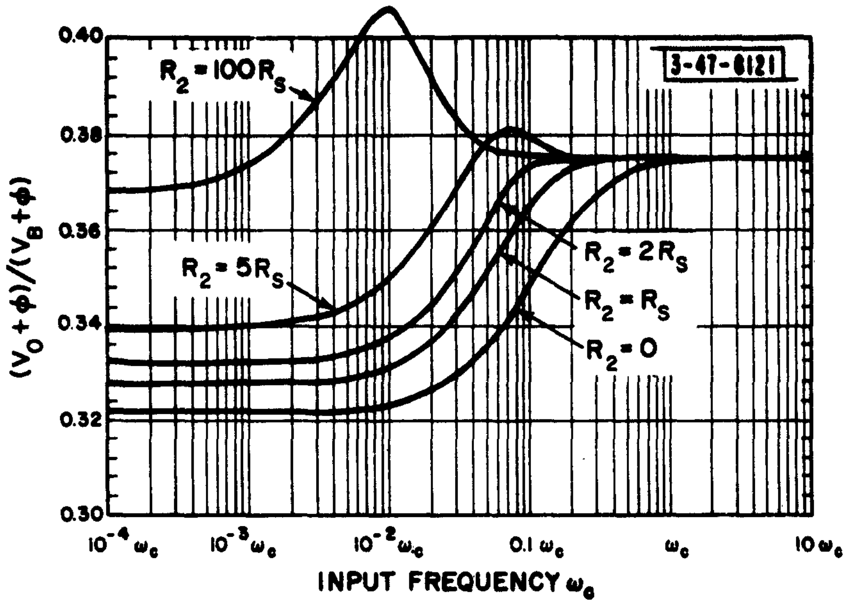

Fig. 3.8 Bias voltage of a 1-2-3 abrupt-junction-varactor tripler, adjusted for maximum efficiency operation, for various values of idler resistance, R_2.

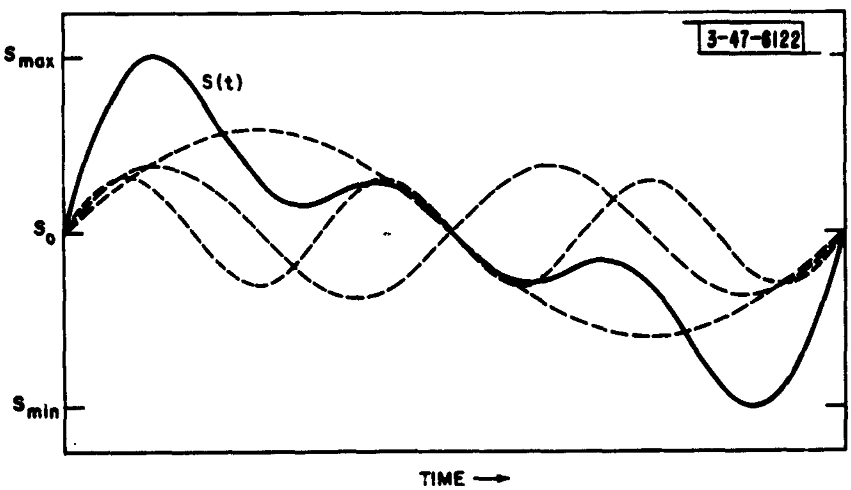

Fig. 3.9 Elastance waveform of an abrupt-junction-varactor tripler at low frequencies, adjusted to yield maximum efficiency with a lossless idler termination ($m_1 = 0.148$, $m_2 = 0.092$, $m_3 = 0.074$). We show the fundamental, second harmonic, and third harmonic individually and their sum.

At low frequencies, we expect the efficiency to approach 100 per cent, since the varactor losses become relatively less important as the reactance gets large. Figure 3. 2 shows that this is indeed the case. In fact, the efficiency even approaches 100 per cent with a nonzero idler resistance. Note also that the highest efficiency occurs with a lossless idler circuit, which is expected from detailed consideration of the properties of idler circuits. At low frequencies the input and load resistances become large. However, at high frequencies the series resistance of the varactor dominates, and the input and load resistances approach R_s.

Input and output power are approximately equal at low frequencies, and vary linearly with frequency, as shown in Figs. 3. 5 and 3. 6. At high frequencies the efficiency is low and, consequently, the input and dissipated powers are almost equal. (In fact, most of the input power is dissipated in the varactor at the fundamental frequency.)

In the figures we have neglected the factor containing S_{min} in the multiplier equations. If S_{min} is not negligible, the values shown in Figs. 3. 5 to 3. 7 must be multiplied by $(S_{max} - S_{min})^2/(S_{max} + S_{min})^2$, i. e., P_{norm} must be replaced by P'_{norm}. The bias voltage as given in Fig. 3. 8 must be modified according to Eq. (3. 11), if S_{min} is not negligible. The operating conditions specified above lead to an average elastance given by

$$S_o = \frac{S_{max} + S_{min}}{2} \quad . \tag{3.19}$$

This is the value of elastance that must be tuned out at each frequency according to the theory.

The conditions for maximum power output are qualitatively very similar to those for maximum efficiency. For small values of idler resistance, R_2, the results are not significantly different from those presented in Figs. 3. 2 to 3. 8 for maximum efficiency. The two optimizations differ quite a bit for large values of R_2, but we do not plot the results because large idler resistances are of little or no practical interest.

It is interesting to examine the elastance waveform under typical operation. The plot shown in Fig. 3.9 is for the low frequency values of m_1, m_2, and m_3 for maximum efficiency operation with a lossless idler. For maximum drive operation, S_{min} and S_{max} are attained once per cycle. At higher frequencies m_2 and m_3 become small, while m_1 increases toward 0.25. Thus, the fundamental elastance dominates the elastance waveform at high frequencies.

3.3 Asymptotic Formulas for the Tripler

At low and high frequencies the tripler behavior can be described by asymptotic formulas. The limiting values of m_1, m_2, and m_3 for low frequencies can be found from the computed data. Then the appropriate formulas are found from Eqs. (3.3) to (3.11). For high frequencies we use the limiting values, $R_{in} \approx R_3 \approx R_s$ and $m_1 \approx 0.25$ ($m_k << m_1$ for $k > 1$), in Eqs. (3.3) to (3.11) to find asymptotic relations. These formulas are summarized in Table 3.1 for the lossless idler case. For low frequencies both maximum power output and maximum efficiency formulas are given. Power output and efficiency are simultaneously maximized at high frequencies, so only one set of formulas is required.

Table 3.1 Asymptotic Formulas for the 1-2-3 Tripler

Maximum power output and maximum efficiency are achieved with a lossless idler, so we have set $R_2 = 0$. For simplicity we have also assumed $S_{min} \ll S_{max}$.

	Low Frequency		High Frequency
	Maximum ϵ	Maximum P_{out}	Max. ϵ and P_{out}
ϵ	$1 - 34.8\frac{\omega_o}{\omega_c}$	$1 - 35.1\frac{\omega_o}{\omega_c}$	$6.08 \times 10^{-5}(\frac{\omega_c}{\omega_o})^4$
R_{in}	$0.137(\frac{\omega_c}{\omega_o})R_s$	$0.126(\frac{\omega_c}{\omega_o})R_s$	R_s
R_3	$0.184(\frac{\omega_c}{3\omega_o})R_s$	$0.168(\frac{\omega_c}{3\omega_o})R_s$	R_s
$\frac{P_{in}}{P_{norm}}$	$0.0241(\frac{\omega_o}{\omega_c})$	$0.0242(\frac{\omega_o}{\omega_c})$	$0.500(\frac{\omega_o}{\omega_c})^2$
$\frac{P_{out}}{P_{norm}}$	$0.0241(\frac{\omega_o}{\omega_c})$	$0.0242(\frac{\omega_o}{\omega_c})$	$3.04 \times 10^{-5}(\frac{\omega_c}{\omega_o})^2$
$\frac{P_{diss}}{P_{norm}}$	$0.837(\frac{\omega_o}{\omega_c})^2$	$0.849(\frac{\omega_o}{\omega_c})^2$	$0.500(\frac{\omega_o}{\omega_c})^2$
$\frac{V_o + \varphi}{V_B + \varphi}$	0.321	0.324	0.375
m_1	0.148	0.155	0.250
m_2	0.091	0.084	$0.0156(\frac{\omega_c}{\omega_o})$
m_3	0.074	0.0775	$0.00065(\frac{\omega_c}{\omega_o})^2$

IV. QUADRUPLER SOLUTIONS

The quadrupler is the only abrupt-junction-varactor multiplier, other than the tripler, which can be made with a single idler (at the second harmonic). As with the tripler, however, there are several alternate ways to construct a quadrupler with two or more idlers. The simplest, multiple-idler quadrupler is one with idlers at $2\omega_o$ and $3\omega_o$. Both the 1-2-4 quadrupler and the 1-2-3-4 quadrupler will be studied. Some other multiple-idler configuration may be better than either of these, but we will not pursue the problem.

The analysis of each quadrupler is very similar to that of the tripler. The techniques of solution and the general nature of the results are the same, although the formulas and specific curves are different.

4.1 1-2-4 Quadrupler Formulas

The formulas of Chapter II apply to this case with all M_k zero except for k = -4, -2, -1, 0, 1, 2, and 4. The idler and load resistance equations become, from Eq. (2.24),

$$\frac{R_4 + R_s}{R_s} = \frac{\omega_c}{8\omega_o} \frac{(jM_2)^2}{jM_4} , \tag{4.1}$$

$$\frac{R_2 + R_s}{R_s} = \frac{\omega_c}{4\omega_o} \frac{(jM_1)^2 - 2(jM_2)^*(jM_4)}{jM_2} . \tag{4.2}$$

The arbitrary phase reference (time origin) may be chosen such that jM_1 is real and positive and thus equal to its magnitude, m_1. Equation (4.1) shows that the phase angle of jM_4 is twice that of jM_2. Finally, the use of this information in Eq. (4.2) demonstrates that jM_2 (and therefore jM_4) must be real and positive; thus, $jM_2 = m_2$ and $jM_4 = m_4$. In terms of m_1, m_2, and m_4 the various 1-2-4 quadrupler formulas are

$$R_4 = R_s\left(\frac{\omega_c}{8\omega_o} \frac{m_2^2}{m_4} - 1\right) , \tag{4.3}$$

$$R_2 = R_s\left(\frac{\omega_c}{4\omega_o} \frac{m_1^2 - 2m_2m_4}{m_2} - 1\right) , \qquad (4.4)$$

$$R_{in} = R_s\left(\frac{\omega_c}{\omega_o} m_2 + 1\right) , \qquad (4.5)$$

$$\frac{P_{in}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 \left(\frac{\omega_c}{\omega_o} m_1^2 m_2 + m_1^2\right) , \qquad (4.6)$$

$$\frac{P_{out}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 \left(\frac{2\omega_c}{\omega_o} m_2^2 m_4 - 16m_4^2\right) , \qquad (4.7)$$

$$\frac{P_{diss}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 \left[\frac{\omega_c}{\omega_o}(m_1^2 m_2 - 2m_2^2 m_4) + m_1^2 + 16m_4^2\right] , \qquad (4.8)$$

$$\frac{P_{diss,v}}{P_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 (m_1^2 + 4m_2^2 + 16m_4^2) , \qquad (4.9)$$

$$\epsilon = \frac{2m_4}{m_1^2} \frac{\frac{\omega_c}{\omega_o} m_2^2 - 8m_4}{\frac{\omega_c}{\omega_o} m_2 + 1} , \qquad (4.10)$$

$$\frac{V_o + \varphi}{V_B + \varphi} = \left(\frac{S_{max} - S_{min}}{S_{max}}\right)^2 \left[\frac{1}{4} + 2(m_1^2 + m_2^2 + m_4^2)\right] + \frac{S_{min}}{S_{max}} , \qquad (4.11)$$

where we have set $m_o = \frac{1}{2}$. The power relations have been normalized with respect to P'_{norm} as defined by Eq. (3.12).

The M_k have been shown to be entirely imaginary. Therefore, Condition (2.17) gives the limit on the magnitudes of the m_k. For the 1-2-4 quadrupler we have

$$m_1 \sin \omega_o t + m_2 \sin 2\omega_o t + m_4 \sin 4\omega_o t \le 0.25 \tag{4.12}$$

for all values of t.

4.2 Solution of the 1-2-4 Quadrupler Equations

The above formulas are written in terms of varactor parameters (R_s, ω_c, P'_{norm}) and the m_k. Hence, if we know values of m_1, m_2, and m_4 which satisfy Condition (4.12) and which give positive values for R_2 and R_4, it is apparent that a few simple calculations will yield the remaining multiplier parameters.

Usually, however, we must find the values of R_2 and R_4 which will maximize either the efficiency or the power handling capability of the quadrupler. Both of these maxima occur when the varactor is optimally driven, that is, with the elastance attaining the values of S_{min} and S_{max} at least once during each cycle. Therefore, the values of the m_k must be such that Condition (4.12) is satisfied with the equality sign at some time t_o when m(t) is a maximum.

As was the case with the tripler, we have to use an iterative numerical procedure to find solutions of the nonlinear quadrupler equations. These calculations are usually performed by choosing values for R_2, R_4, and ω_o/ω_c and then computing the required values for m_1, m_2, and m_4. It is convenient, therefore, to solve Eqs. (4.3) and (4.4) for m_1 and m_4 in terms of m_2 (this choice of reference is found to give the simplest equations):

$$\left(\frac{m_1}{m_2}\right)^2 = \frac{R_2 + R_s}{R_s} \frac{4\omega_o}{m_2 \omega_c} + \frac{R_s}{R_4 + R_s} \frac{m_2 \omega_c}{4\omega_o} \quad , \tag{4.13}$$

$$\frac{m_4}{m_2} = \frac{m_2 \omega_c}{8\omega_o} \frac{R_s}{R_4 + R_s} \quad . \tag{4.14}$$

The numerical procedure used for solving the quadrupler equations is exactly the same as described in Section 3.2 for the tripler, except that in this case m_2 is the control parameter. Computations for maximum

efficiency operation and maximum power output operation were performed on an I. B. M. 7090 digital computer. The results for maximum efficiency operation are given in Figs. 4. 1 to 4. 7. Efficiency, input resistance, load resistance, power input, power output, dissipated power, and bias voltage are plotted as functions of frequency for several values of R_2.

For a given frequency and a specific value of R_2, the efficiency, the power input, and the power output as functions of the load resistance look somewhat like those shown in Fig. 3. 1 for the tripler. The three maxima usually occur at slightly different values of R_4 at low frequencies. However, the efficiency is not much reduced where power output is a maximum, and vice versa. Thus for practical purposes the power output and the efficiency can be maximized simultaneously. At high frequencies both maxima occur with the same load. The maxima are quite broad which explains the reasonably good predictions of the previous quasi-optimum solution.

At low frequencies the efficiency approaches 100 per cent, even with a nonzero idler resistance, as expected. Power input and power output are approximately equal at low frequencies, while at high frequencies input and dissipated powers are nearly equal. Also, at high frequencies the input and load resistances become approximately equal to R_s.

In the figures we have neglected the factors containing S_{min} in Eqs. (4. 6), (4. 7), (4. 8), and (4. 11). If S_{min} is not negligible, we must use P'_{norm} instead of P_{norm} for the normalization power, and the bias voltage as given in Fig. 4. 7 must be modified according to Eq. (4. 11). The average elastance is given by Eq. (3. 19), since our solutions are for maximum drive.

The conditions for maximum power output are very similar to those for maximum efficiency. For small values of idler resistance, the results are not significantly different from those presented in Figs. 4. 1, 4. 4, 4. 5, and 4. 6. The input and load resistances and the bias voltage are somewhat different, but not enough to make replotting necessary. The two optimizations differ quite a bit for large values of R_2; however, we do not plot the results because in practice large idler resistances are of little or no interest.

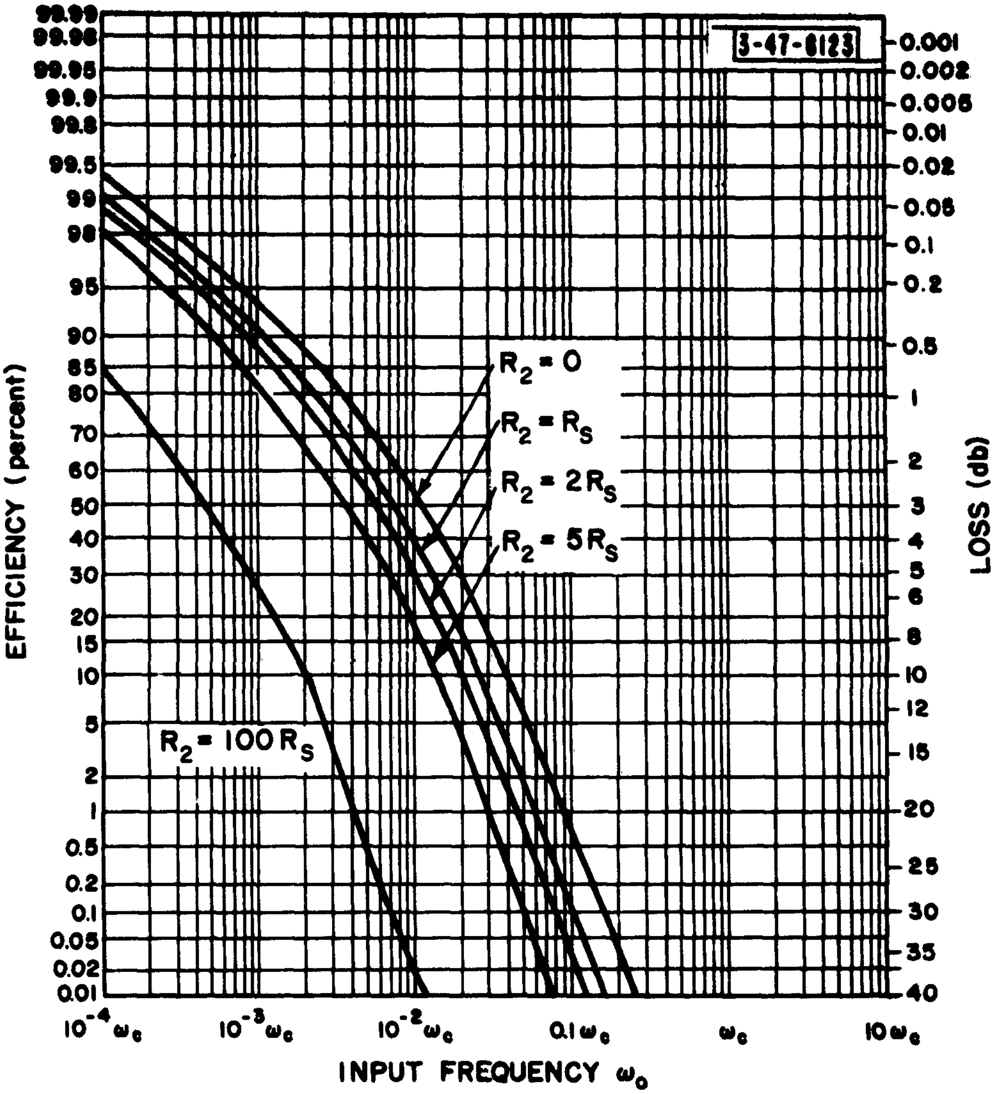

Fig. 4.1 Maximum efficiency of a 1-2-4 abrupt-junction-varactor quadrupler for a variety of idler resistances, R_2.

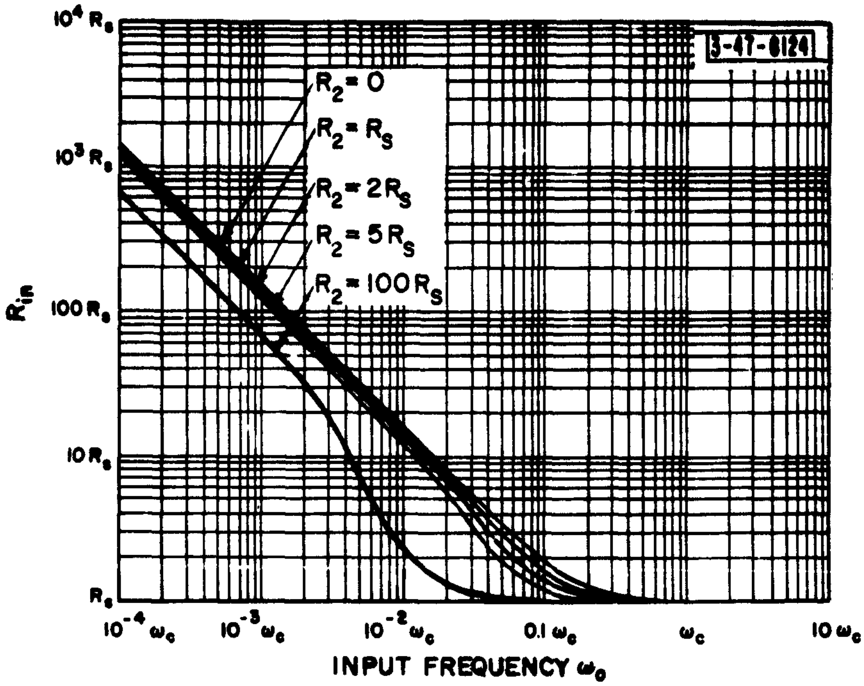

Fig. 4.2 Input resistance of a 1-2-4 abrupt-junction-varactor quadrupler adjusted for maximum efficiency operation, for various values of idler resistance, R_2.

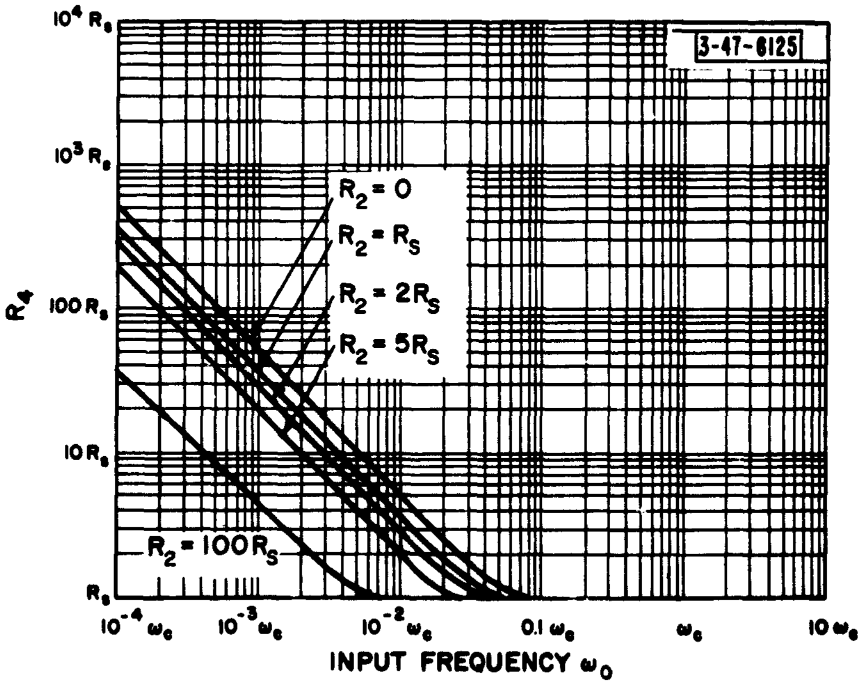

Fig. 4.3 Load resistance for a 1-2-4 abrupt-junction-varactor quadrupler for maximum efficiency operation.

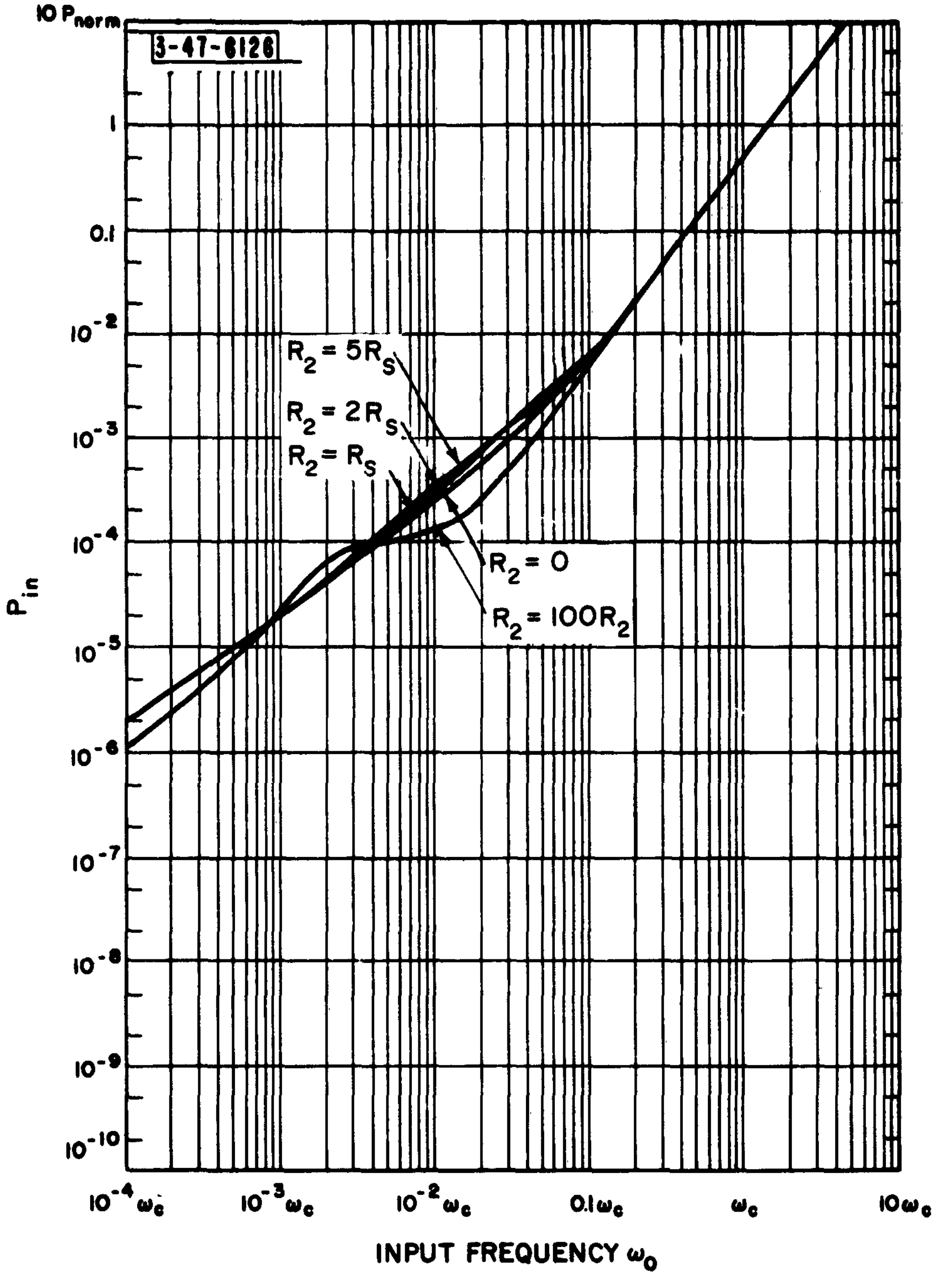

Fig. 4.4 Input power of a 1-2-4 abrupt-junction-varactor quadrupler, adjusted for maximum efficiency operation for various values of idler resistance, R_2.

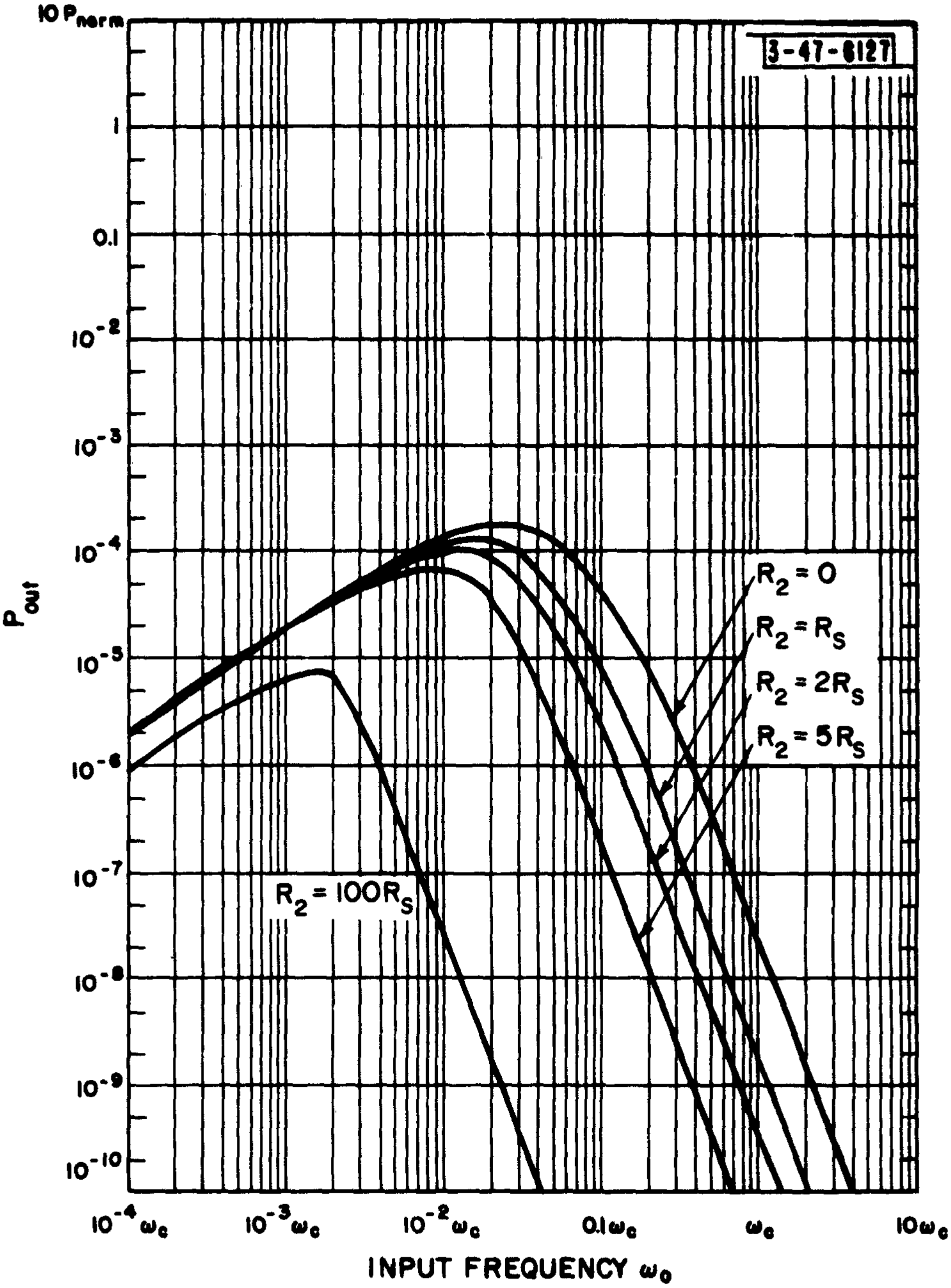

Fig. 4.5 Output power for a 1-2-4 abrupt-junction-varactor quadrupler for maximum efficiency operation.

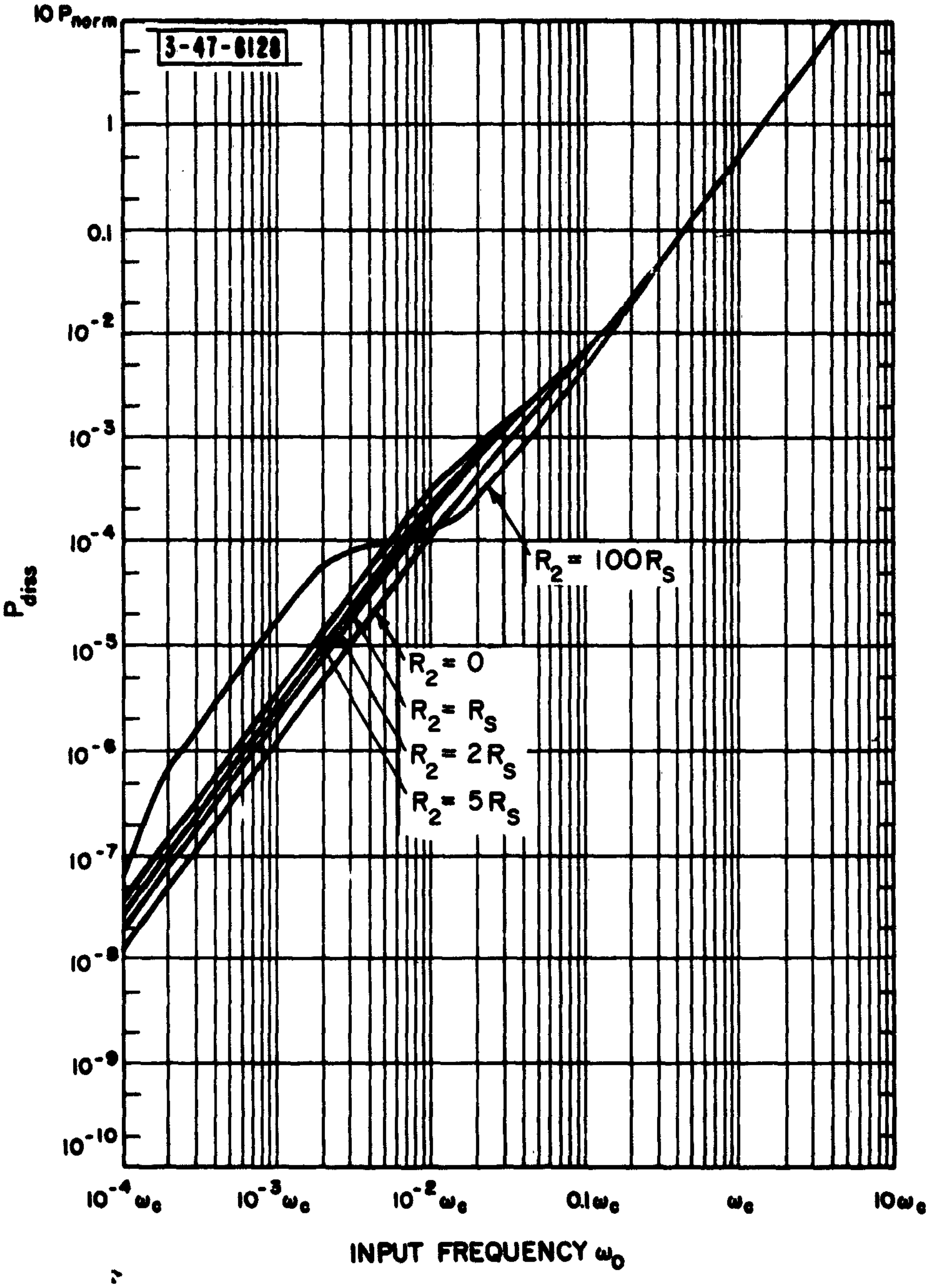

Fig. 4.6 Total dissipated power (power dissipated in the varactor plus the power dissipated in the idler termination) for a 1-2-4 abrupt-junction-varactor quadrupler, adjusted for maximum efficiency operation, for various values of idler resistance, R_2.

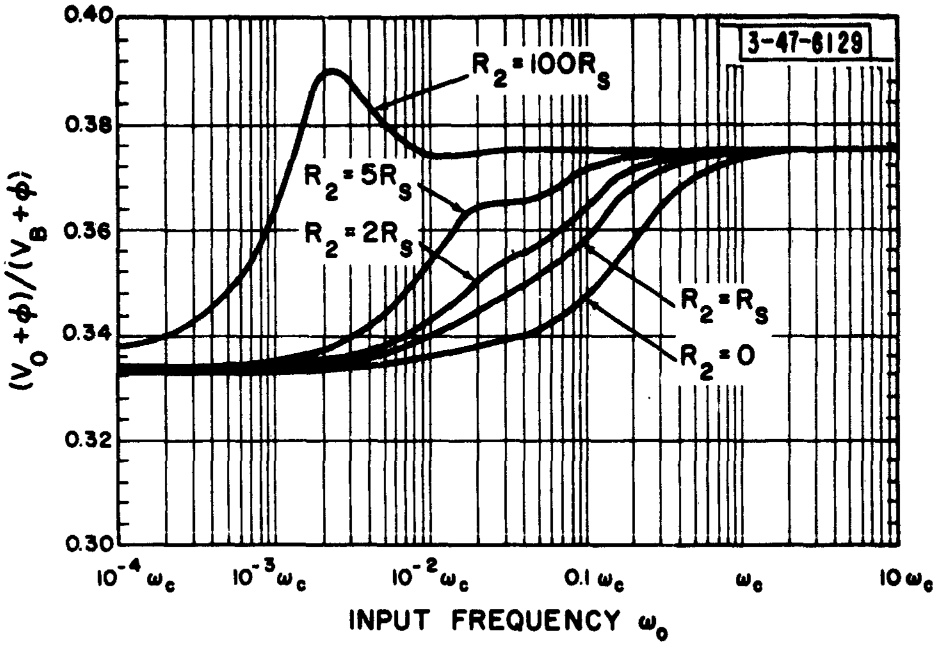

Fig. 4.7 Bias voltage for a 1-2-4 quadrupler adjusted for maximum efficiency operation.

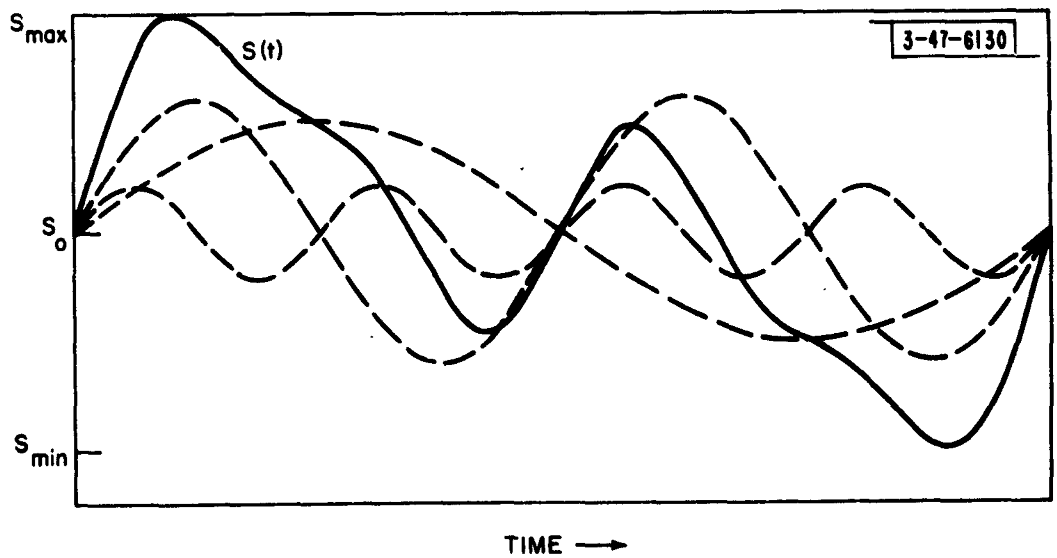

Fig. 4.8 Elastance waveform of a 1-2-4 quadrupler adjusted for maximum efficiency operation. The plot is for low frequencies with a lossless idler (m_1 = 0.13, m_2 = 0.15 and m_4 = 0.053).

A plot of the elastance waveform under typical operation is shown in Fig. 4.8. The low frequency values of m_1, m_2, and m_4 for maximum efficiency operation with a lossless idler are used in the figure. Since the solutions are for maximum drive, the elastance varies between S_{min} and S_{max} during each cycle.

4.3 Asymptotic Formulas for the 1-2-4 Quadrupler

At low and high frequencies the behavior of the 1-2-4 quadrupler can be described by asymptotic formulas. The limiting values of m_1, m_2, and m_4 for low frequencies can be found from the computed data. Then the appropriate formulas are found from Eqs. (4.3) to (4.11). For high frequencies we use the limiting values, $R_{in} \approx R_4 \approx R_s$ and $m_1 \approx 0.25$ ($m_k \ll m_1$ for $k > 1$), in Eqs. (4.3) to (4.11) to find asymptotic relations. These formulas are summarized in Table 4.1 for the lossless idler case. For low frequencies both maximum power output and maximum efficiency formulas are given. Power output and efficiency are simultaneously maximized at high frequencies, so only one set of formulas is required.

4.4 1-2-3-4 Quadrupler Formulas

The idler and load resistance equations for the 1-2-3-4 quadrupler can be written directly from Eq. (2.24) where the M_k are zero except when k takes on the values, -4, -3, -2, -1, 0, 1, 2, 3, and 4. Thus

$$\frac{R_4 + R_s}{R_s} = \frac{\omega_c}{8\omega_o} \frac{(jM_2)^2 + 2(jM_1)(jM_3)}{jM_4} , \tag{4.15}$$

$$\frac{R_3 + R_s}{R_s} = \frac{\omega_c}{3\omega_o} \frac{(jM_1)(jM_2) - (jM_1)^*(jM_4)}{jM_3} , \tag{4.16}$$

$$\frac{R_2 + R_s}{R_s} = \frac{\omega_c}{4\omega_o} \frac{(jM_1)^2 - 2(jM_1)^*(jM_3) - 2(jM_2)^*(jM_4)}{jM_2} . \tag{4.17}$$

The addition of the extra idler considerably complicates the problem of finding phase relationships. However, by selecting the time origin such

Table 4.1 Asymptotic Formulas for the 1-2-4 Quadrupler

Maximum power output and maximum efficiency are achieved with a lossless idler, so we have set $R_2 = 0$. For simplicity we have also assumed $S_{min} << S_{max}$.

	Low Frequency		High Frequency
	Maximum ϵ	Maximum P_{out}	Max. ϵ and P_{out}
ϵ	$1 - 62.5(\frac{\omega_o}{\omega_c})$	$1 - 66.2(\frac{\omega_o}{\omega_c})$	$5.96 \times 10^{-8}(\frac{\omega_c}{\omega_o})^6$
R_{in}	$0.150(\frac{\omega_c}{\omega_o})R_s$	$0.136(\frac{\omega_c}{\omega_o})R_s$	R_s
P_4	$0.205(\frac{\omega_c}{4\omega_o})R_s$	$0.136(\frac{\omega_c}{4\omega_o})R_s$	R_s
$\frac{P_{in}}{P_{norm}}$	$0.0196(\frac{\omega_o}{\omega_c})$	$0.0201(\frac{\omega_o}{\omega_c})$	$0.500(\frac{\omega_o}{\omega_c})^2$
$\frac{P_{out}}{P_{norm}}$	$0.0196(\frac{\omega_o}{\omega_c})$	$0.0201(\frac{\omega_o}{\omega_c})$	$2.98 \times 10^{-8}(\frac{\omega_c}{\omega_o})^4$
$\frac{P_{diss}}{P_{norm}}$	$1.23(\frac{\omega_o}{\omega_c})^2$	$1.33(\frac{\omega_o}{\omega_c})^2$	$0.500(\frac{\omega_o}{\omega_c})^2$
$\frac{V_o + \varphi}{V_B + \varphi}$	0.334	0.334	0.375
m_1	0.128	0.136	0.250
m_2	0.150	0.136	$0.0156(\frac{\omega_c}{\omega_o})$
m_4	0.055	0.068	$1.53 \times 10^{-5}(\frac{\omega_c}{\omega_o})^3$

that jM_1 is real and positive and thus equal to its magnitude, m_1, we can write Eqs. (4.15), (4.16), and (4.17) as

$$\frac{m_2}{m_1} e^{j\phi_2} = a \left[1 - 2\frac{m_3}{m_1} e^{j\phi_3} - 2\frac{m_2}{m_1}\frac{m_4}{m_1} e^{j(\phi_4 - \phi_2)}\right], \tag{4.18}$$

$$\frac{m_3}{m_1} e^{j\phi_3} = b \left[\frac{m_2}{m_1} e^{j\phi_2} - \frac{m_4}{m_1} e^{j\phi_4}\right], \tag{4.19}$$

$$\frac{m_4}{m_1} e^{j\phi_4} = c \left[\left(\frac{m_2}{m_1}\right)^2 e^{j2\phi_2} + 2\frac{m_3}{m_1} e^{j\phi_3}\right], \tag{4.20}$$

where we have set $jM_k = m_k e^{j\phi_k}$, $a = \frac{m_1\omega_c}{4\omega_o}\frac{R_s}{R_2 + R_s}$, $b = \frac{m_1\omega_c}{3\omega_o}\frac{R_s}{R_3 + R_s}$, and $c = \frac{m_1\omega_c}{8\omega_o}\frac{R_s}{R_4 + R_s}$. Equations (4.19) and (4.20) are now solved for m_3/m_1 and m_4/m_1 in terms of m_2/m_1

$$\frac{m_4}{m_1} e^{j\phi_4} = A\left(1 + \frac{1}{2b}\frac{m_2}{m_1} e^{j\phi_2}\right)\frac{m_2}{m_1} e^{j\phi_2}, \tag{4.21}$$

$$\frac{m_3}{m_1} e^{j\phi_3} = \frac{A}{2}\left(\frac{1}{c} - \frac{m_2}{m_1} e^{j\phi_2}\right)\frac{m_2}{m_1} e^{j\phi_2}, \tag{4.22}$$

where

$$A = \frac{2bc}{1 + 2bc}. \tag{4.23}$$

Equations (4.21) and (4.22) can be used in Eq. (4.18) to obtain

$$A\left(\frac{m_2}{m_1}\right)^2 \left(2 - e^{j2\phi_2}\right) + \left[\frac{1}{a} + \frac{A}{c} + \frac{A}{b}\left(\frac{m_2}{m_1}\right)^2\right]\frac{m_2}{m_1} e^{j\phi_2} - 1 = 0. \tag{4.24}$$

Equation (4.24) can be separated into two equations relating real and imaginary parts when the relation, $e^{j\Theta} = \cos\Theta + j\sin\Theta$, is used. Thus, we have

$$A\left(\frac{m_2}{m_1}\right)^2 (2 - \cos 2\phi_2) + \left[\frac{1}{a} + \frac{A}{c} + \frac{A}{b}\left(\frac{m_2}{m_1}\right)^2\right] \frac{m_2}{m_1} \cos\phi_2 - 1 = 0. \quad (4.2$$

and

$$\sin\phi_2 \left[\frac{1}{a} + \frac{A}{c} + \frac{A}{b}\left(\frac{m_2}{m_1}\right)^2 - 2A\left(\frac{m_2}{m_1}\right)\cos\phi_2\right] = 0 \ , \quad (4.26)$$

where the trigometric identity, $\sin 2\Theta = 2\sin\Theta\cos\Theta$, has been used to write Eq. (4.26) in the form given. Equation (4.26) is obviously satisfied for $\phi_2 = n\pi$, $n = 0, 1, 2, \ldots$ Only the cases $\phi_2 = 0$ and π need to be investigated, since the same results are obtained for $n \geq 2$. For $\phi_2 = 0$, Eq. (4.25) becomes

$$\frac{A}{b}\left(\frac{m_2}{m_1}\right)^3 + A\left(\frac{m_2}{m_1}\right)^2 + \left(\frac{1}{a} + \frac{A}{c}\right)\frac{m_2}{m_1} - 1 = 0 \ . \quad (4.27)$$

According to Descartes' rule of signs*, Eq. (4.27) has exactly one positive real root regardless of the values of the coefficients (the coefficients are, of course, positive and real). It is this root which we seek as our solution for m_2/m_1.

For the alternate posibility, $\phi_2 = \pi$, Eq. (4.25) becomes

$$\frac{A}{b}\left(\frac{m_2}{m_1}\right)^3 - A\left(\frac{m_2}{m_1}\right)^2 + \left(\frac{1}{a} + \frac{A}{c}\right)\frac{m_2}{m_1} + 1 = 0 \ . \quad (4.28)$$

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*See any standard text on college algebra which covers the theory of equations. For example, M. Richardson, "College Algebra", Prentice-Hall Inc., New York, N. Y., 1947.

We again apply Descartes' rule of signs to find that Eq. (4.28) has either two or zero positive real roots. Therefore, a solution for $\phi_2 = \pi$ may or may not exist depending on the values of the coefficients in Eq. (4.28). In particular, it can be shown that no roots exist at frequencies (ω_o/ω_c) of the order of 0.1 or greater. This point will not be pursued further, since it is later found that a properly optimized multiplier will operate in just the single mode, $\phi_2 = 0$.

There is a third possible solution of Eq. (4.26) which gives a value for ϕ_2 which is not equal to an integer multiple of π. The appropriate solution is

$$\cos\phi_2 = \frac{\frac{1}{a} + \frac{A}{c} + \frac{A}{b}\left(\frac{m_2}{m_1}\right)^2}{2A\left(\frac{m_2}{m_1}\right)} . \tag{4.29}$$

This solution will only be possible when the right hand side of Eq. (4.29) lies between zero and one. It is later found that a, b, and c for an optimized multiplier take on values such that the right hand side of Eq. (4.29) is always greater than one. Therefore, we will usually be able to neglect Eq. (4.29) as a possible solution.

For any of the above cases we can write the following formulas for the 1-2-3-4 quadrupler:

$$R_4 = R_s\left(\frac{\omega_c}{8\omega_o}\ \frac{m_2^2 e^{j2\phi_2} + 2m_1m_3e^{j\phi_3}}{m_4e^{j\phi_4}} - 1\right) , \tag{4.30}$$

$$R_3 = R_s\left(\frac{\omega_c}{3\omega_o}\ \frac{m_1m_2e^{j\phi_2} - m_1m_4e^{j\phi_4}}{m_3e^{j\phi_3}} - 1\right) , \tag{4.31}$$

$$R_2 = R_s\left(\frac{\omega_c}{4\omega_o}\ \frac{m_1^2 - 2m_1m_3\, e^{j\phi_3} - 2m_2m_4\, e^{j(\phi_4 - \phi_2)}}{m_2\, e^{j\phi_2}} - 1\right), \tag{4.32}$$

$$R_{in} = R_s\left\{1 + \frac{\omega_c}{\omega_o}\, \mathrm{Re}\left[\frac{m_1m_2\, e^{j\phi_2} + m_2m_3\, e^{j(\phi_3 - \phi_2)} + m_3m_4\, e^{j(\phi_4 - \phi_3)}}{m_1}\right]\right. \tag{4.33}$$

$$\frac{P_{in}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 m_1^2\ \frac{R_{in}}{R_s}\ , \tag{4.34}$$

$$\frac{P_{out}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 16m_4^2\ \frac{R_4}{R_s}\ , \tag{4.35}$$

$$\frac{P_{diss}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 \left[m_1^2 + 4m_2^2\left(1 + \frac{R_2}{R_s}\right) + 9m_3^2\left(1 + \frac{R_3}{R_s}\right) + 16m_4^2\right], \tag{4.36}$$

$$\frac{P_{diss,v}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 (m_1^2 + 4m_2^2 + 9m_3^2 + 16m_4^2)\ , \tag{4.37}$$

$$\epsilon = \frac{16m_4^2}{m_1^2} \cdot \frac{R_4}{R_{in}}\ , \tag{4.38}$$

$$\frac{V_o + \varphi}{V_B + \varphi} = \left(\frac{S_{max} - S_{min}}{S_{max}}\right)^2 \left[\left(m_o + \frac{S_{max}}{S_{max} - S_{min}}\right)^2 + 2(m_1^2 + m_2^2 + m_3^2 + m_4^2)\right]\ , \tag{4.39}$$

where the power relations have been normalized with respect to P'_{norm} as defined by Eq. (3.12).

In general, the jM_k are complex numbers, so we must use Condition (2.15) rather than the special one given by Eq. (2.17). Thus we have

$$0 \le m_o + 2m_1 \sin \omega_o t + 2m_2 \sin (2\omega_o t + \phi_2)$$

$$+ 2m_3 \sin (3\omega_o t + \phi_3) + 2m_4 \sin (4\omega_o t + \phi_4) \le 1 \qquad (4.40)$$

for all values of t.

4.5 Solution of the 1-2-3-4 Quadrupler Equations

To properly treat this multiplier we would have to separately examine each of the possible solutions. Rather than attack the problem in this way, we begin by assuming $\phi_2 = 0$ and then proceed as if this choice yielded the only solution. We know that it yields the only solution at high frequencies. However, at lower frequencies there may be a problem of jumping from one mode of operation to another. If "phase jumps" from mode to mode occurred in practice, the multiplier would probably be useless because of the resulting efficiency and/or power level changes. Therefore, we attack the problem by maximizing efficiency (or power output) for one mode of operation ($\phi_2 = 0$) and then check to see whether any other mode of operation is possible with the specified idler and load resistances. For the 1-2-3-4 quadrupler, it is found that the phase stays locked with $\phi_2 = \phi_3 = \phi_4 = 0$ for maximum efficiency or maximum power output operation. Therefore, this quadrupler will be a useful device when operated with an optimized load resistance.

If a non-optimum load ($R_4 < R_{4,opt}$ only) is used, "phase jumping" may be a very serious problem. This problem has not been investigated to any extent except to show its existence. Probably the phase will remain locked even with load resistances about one-fourth of the optimum value. Since most multiplier designs are based on maximum efficiency or maximum power output operation, "phase jumping" should not be an important problem.

The numerical procedures used for solving the 1-2-3-4 quadrupler equations are the same as described in Section 3.2 for the tripler. In this case (ϕ_2 set equal to zero), Eqs. (4.27), (4.21), and (4.22) are used for the

calculation of m_2, m_3, and m_4 in terms of m_1. The limit on the magnitudes of the m_k is given by Condition (4.40) with $m_o = 1/2$ and $\phi_2 = \phi_3 = \phi_4 = 0$. (For maximum efficiency and maximum power output operation, ϕ_3 is equal to zero, which is not necessarily the case under non-optimum conditions.)

Computations for maximum efficiency operation with $\phi_2 = 0$ were performed on an I.B.M. 7090 digital computer. The results are given in Figs. 4.9 to 4.15. Efficiency, input resistance, load resistance, power input, power output, dissipated power, and bias voltage are plotted as functions of frequency for several values of R_2 and R_3.

The curves in the figures are qualitatively very similar to the tripler and the 1-2-4 quadrupler curves and are interpreted in the same way. In the figures we have neglected the factors containing S_{min}. If S_{min} is not negligible, we must use P'_{norm} instead of P_{norm} for the normalization power, and the bias voltage as given in Fig. 4.15 must be modified according to Eq. (4.39) with m_o set equal to one-half. The average elastance is given by Eq. (3.19), since the solutions are for maximum drive and $m_o = 1/2$.

A plot of the elastance waveform under typical operation is shown in Fig. 4.16. The low frequency values of m_1, m_2, m_3, and m_4 for maximum efficiency operation with lossless idlers ($R_2 = R_3 = 0$) are used in the figure. For maximum drive operation, the elastance attains the values of S_{min} and S_{max} one or more times during each cycle.

4.6 Asymptotic Formulas for the 1-2-3-4 Quadrupler

At low and high frequencies the behavior of the 1-2-3-4 quadrupler can be described by asymptotic formulas. The limiting values of m_1, m_2, m_3, and m_4 for low frequencies can be found from the computed data. Then the appropriate formulas are found from Eqs. (4.26) to (4.35) with $\phi_2 = \phi_3 = \phi_4 = 0$. For high frequencies we use the limiting values, $R_{in} \approx R_4 \approx R_s$ and $m_1 \approx 0.25$ ($m_k \ll m_1$ for $k > 1$), in Eqs. (4.18) to (4.20) and (4.30) to (4.35) to find asymptotic relations. These formulas are summarized in Table 4.2 for maximum efficiency operation with lossless idlers. The high-frequency asymptotes apply for both maximum efficiency and maximum power output operation.

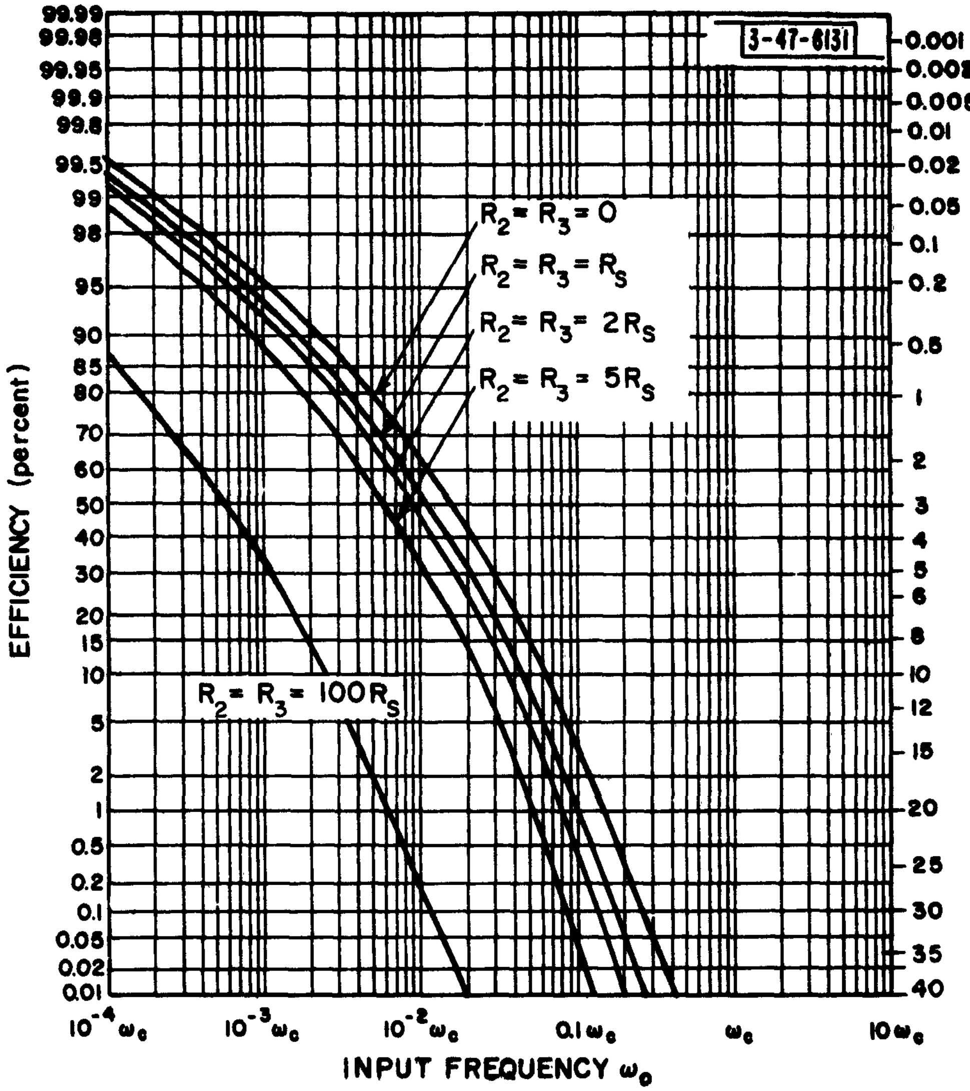

Fig. 4.9 Maximum efficiency of a 1-2-3-4 quadrupler for a variety of idler resistances, R_2 and R_3.

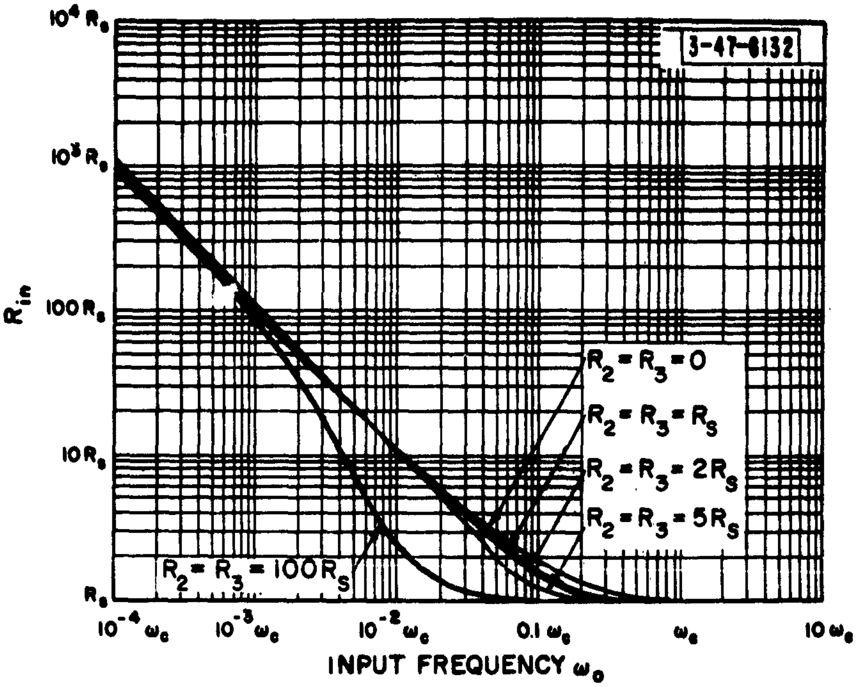

Fig. 4.10 Input resistance for maximum efficiency operation of a 1-2-3-4 quadrupler for several values of idler resistances, R_2 and R_3.

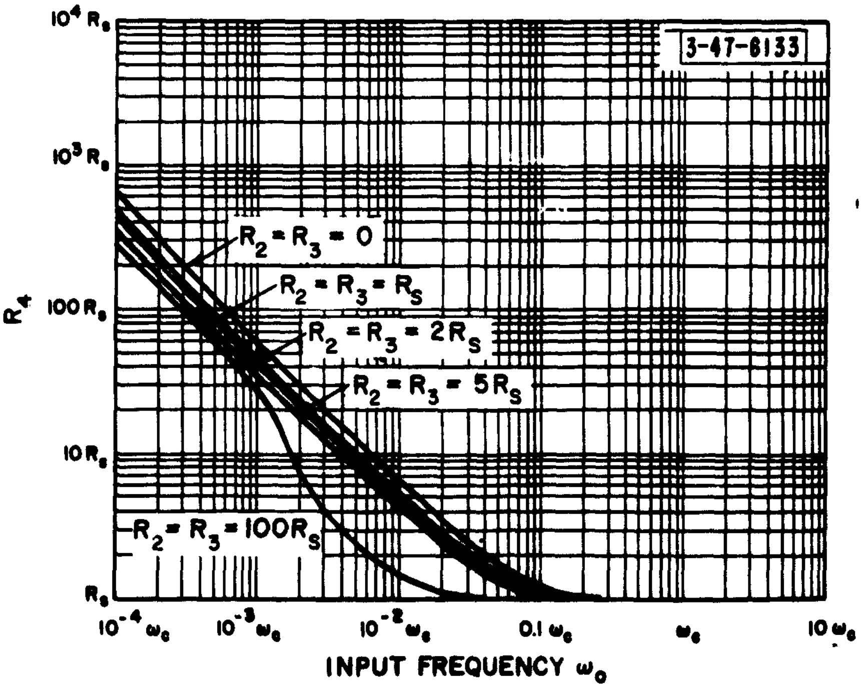

Fig. 4.11 Load resistance for a 1-2-3-4 quadrupler adjusted for maximum efficiency operation.

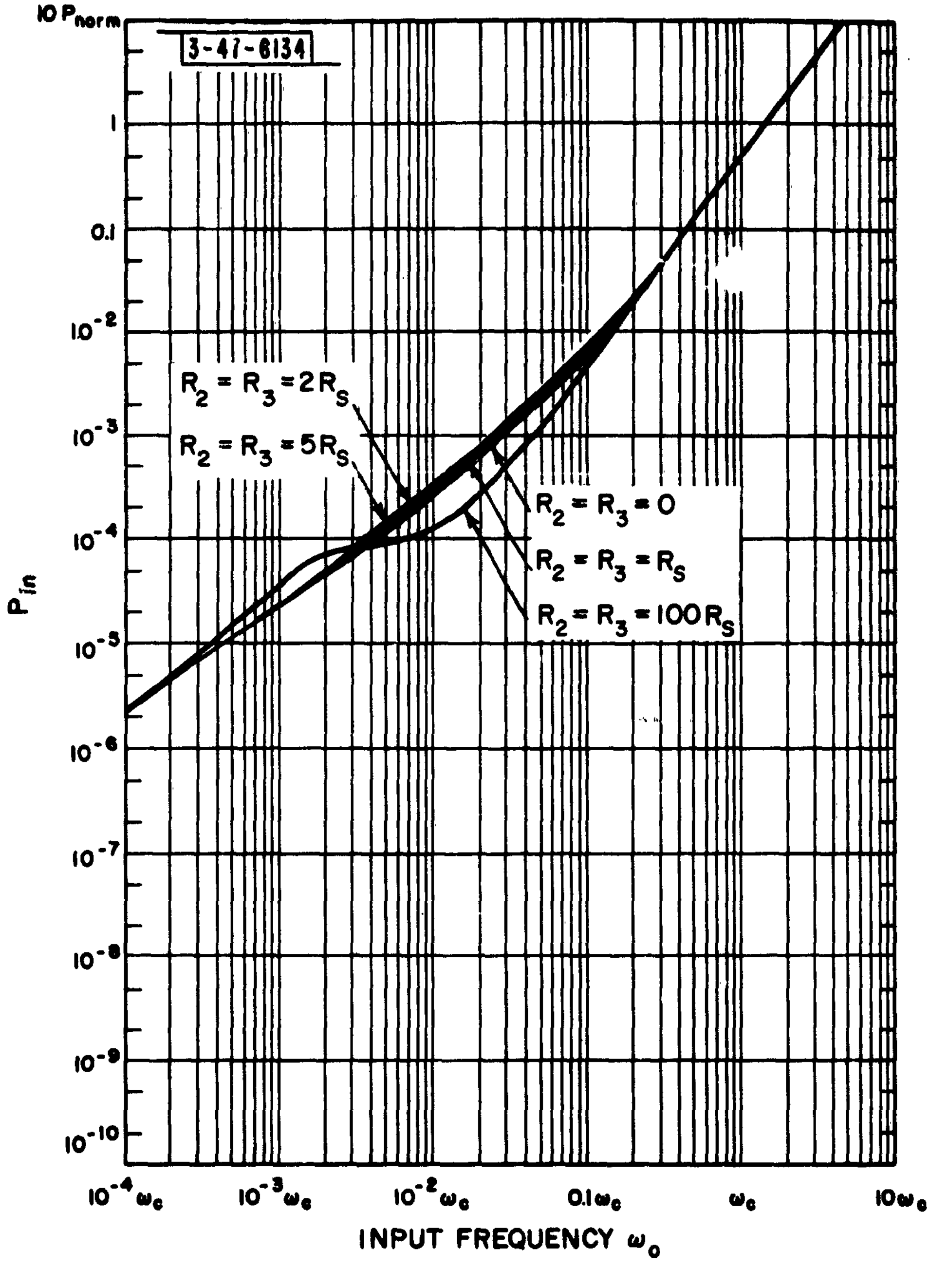

Fig. 4.12 Power input for maximum efficiency operation of a 1-2-3-4 quadrupler for a variety of idler resistances, R_2 and R_3.

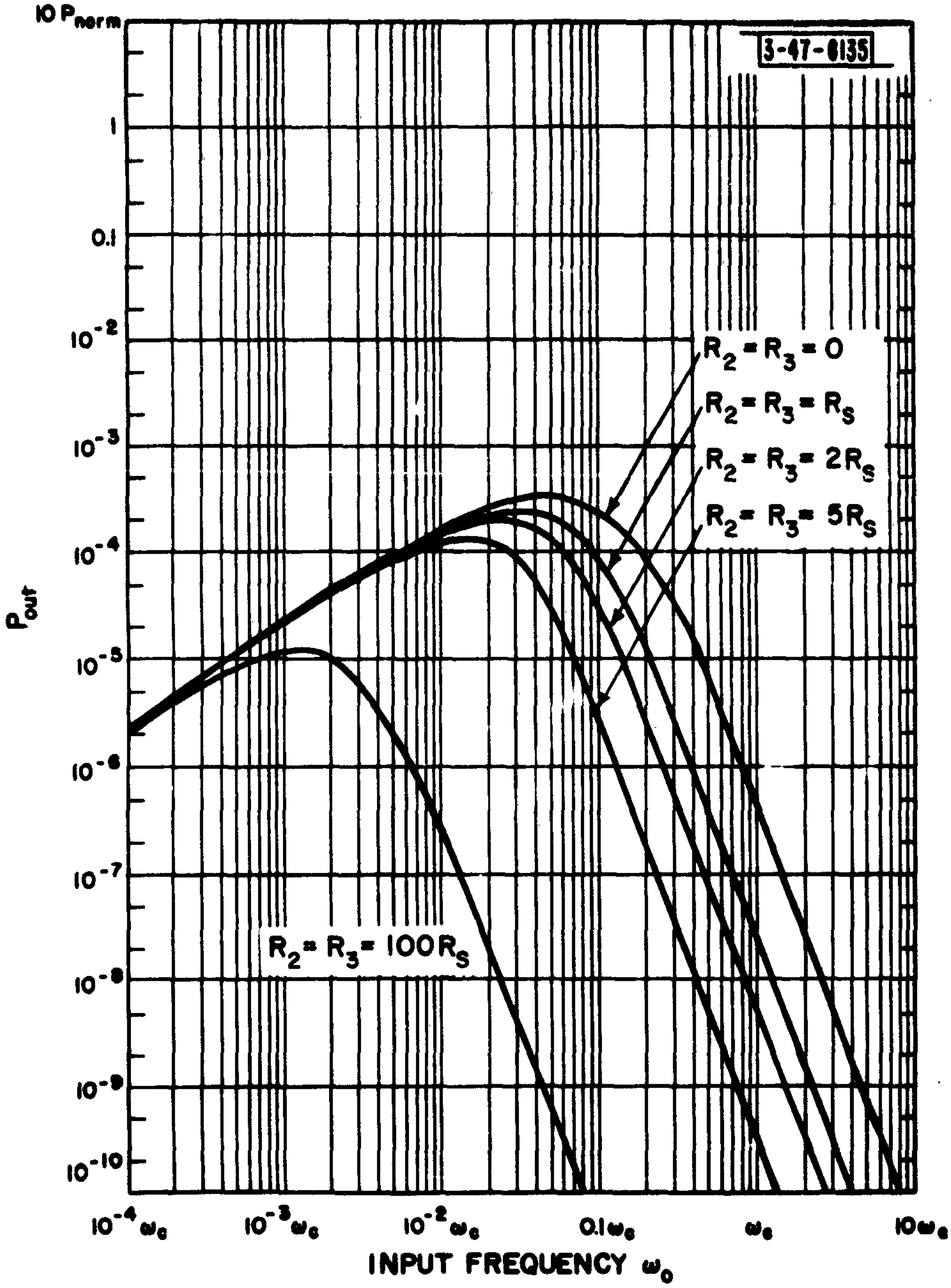

Fig. 4.13 Power output for maximum efficiency operation of a 1-2-3-4 quadrupler for a variety of idler resistances, R_2 and R_3.

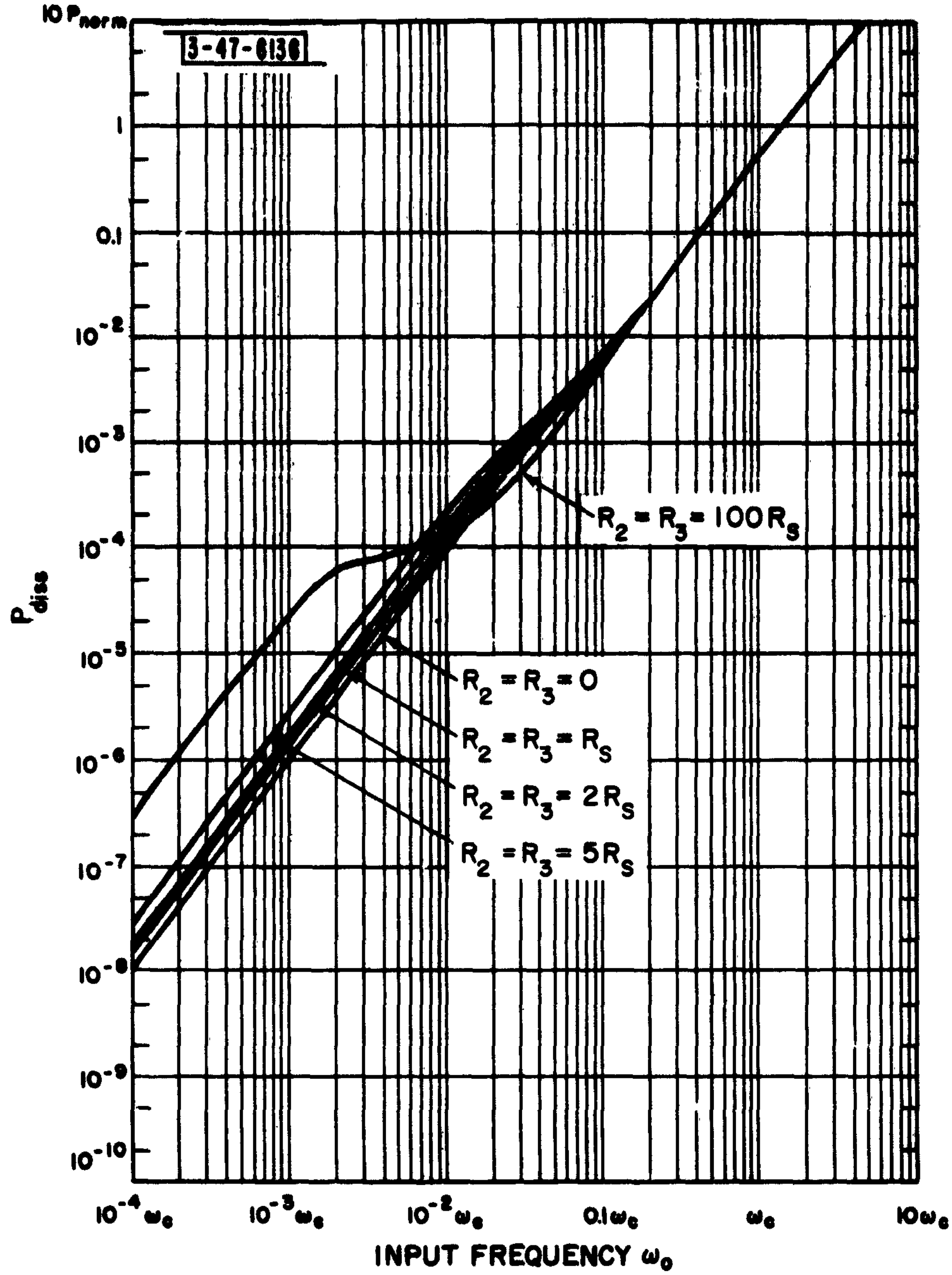

Fig. 4.14 **Power dissipated in both the varactor and the idler terminations for a 1-2-3-4 quadrupler adjusted for maximum efficiency operation.**

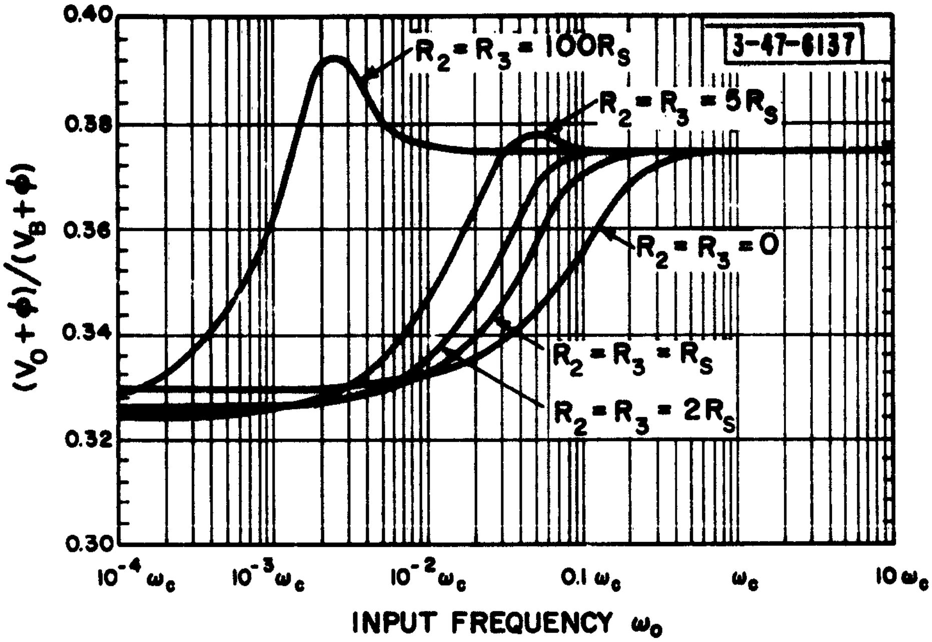

Fig. 4.15 Bias voltage for a 1-2-3-4 quadrupler adjusted for maximum efficiency operation for a variety of idler resistances, R_2 and R_3.

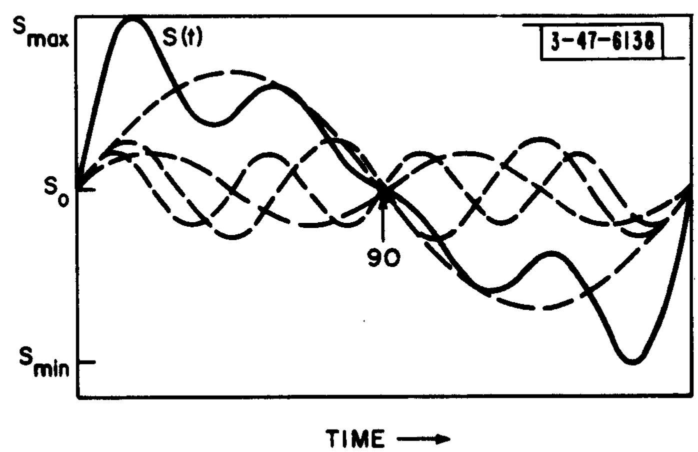

Fig. 4.16 Elastance waveform of a low-frequency 1-2-3-4 quadrupler adjusted for maximum efficiency operation with lossless idler terminations.

Table 4.2 Asymptotic Formulas for the 1-2-3-4 Quadrupler

Low-frequency and high-frequency formulas are given for the abrupt-junction-varactor quadrupler with idlers at $2\omega_o$ and $3\omega_o$. We have assumed that S_{min} is negligible in comparison to S_{max}, and that $R_2 = R_3 = 0$.

	Low-frequency	High-frequency
	Maximum ϵ	Max. ϵ and P_{out}
ϵ	$1 - 45.6(\frac{\omega_o}{\omega_c})$	$8.0 \times 10^{-7}(\frac{\omega_c}{\omega_o})^6$
R_{in}	$.096(\frac{\omega_c}{\omega_o}) R_s$	R_s
R_4	$.250(\frac{\omega_c}{\omega_o}) R_s$	R_s
$\frac{P_{in}}{P_{norm}}$	$0.0226(\frac{\omega_o}{\omega_c})$	$0.500(\frac{\omega_o}{\omega_c})^2$
$\frac{P_{out}}{P_{norm}}$	$0.0226(\frac{\omega_o}{\omega_c})$	$4.0 \times 10^{-7}(\frac{\omega_c}{\omega_o})^4$
$\frac{P_{diss}}{P_{norm}}$	$1.033(\frac{\omega_o}{\omega_c})^2$	$0.500(\frac{\omega_o}{\omega_c})^2$
$\frac{V_o + \varphi}{V_B + \varphi}$	0.330	0.375
m_1	0.1715	0.250
m_2	0.0532	$0.0156(\frac{\omega_c}{\omega_o})$
m_3	0.0693	$0.0013(\frac{\omega_c}{\omega_o})^2$
m_4	0.0532	$5.6 \times 10^{-5}(\frac{\omega_c}{\omega_o})^3$

4.7 Comparison of the 1-2-4 Quadrupler and the 1-2-3-4 Quadrupler

Since two practical idler configurations exist for the abrupt-junction-varactor quadrupler, it is of interest to compare their performance. Most of the comparisons can be made by referring to Figs. 4.1 to 4.7 and Table 4.1 and to Figs. 4.9 to 4.15 and Table 4.2.

The maximum efficiencies with lossless idlers for the two quadruplers are shown in Fig. 4.17 for comparison. Higher efficiency is obtainable from the 1-2-3-4 quadrupler for all frequencies, although the difference is not very large. The maximum difference in efficiency is about 13 per cent.

Also of considerable interest is the comparative power handling capability of the two quadruplers. The power outputs at maximum efficiency with lossless idlers are shown in Fig. 4.18 for comparison. The 1-2-3-4 quadrupler has the highest power output at all frequencies, but the difference is quite small. At low frequencies the ratio of the power outputs, $P_{out}(1\text{-}2\text{-}4)/P_{out}(1\text{-}2\text{-}3\text{-}4)$, is approximately 0.867.

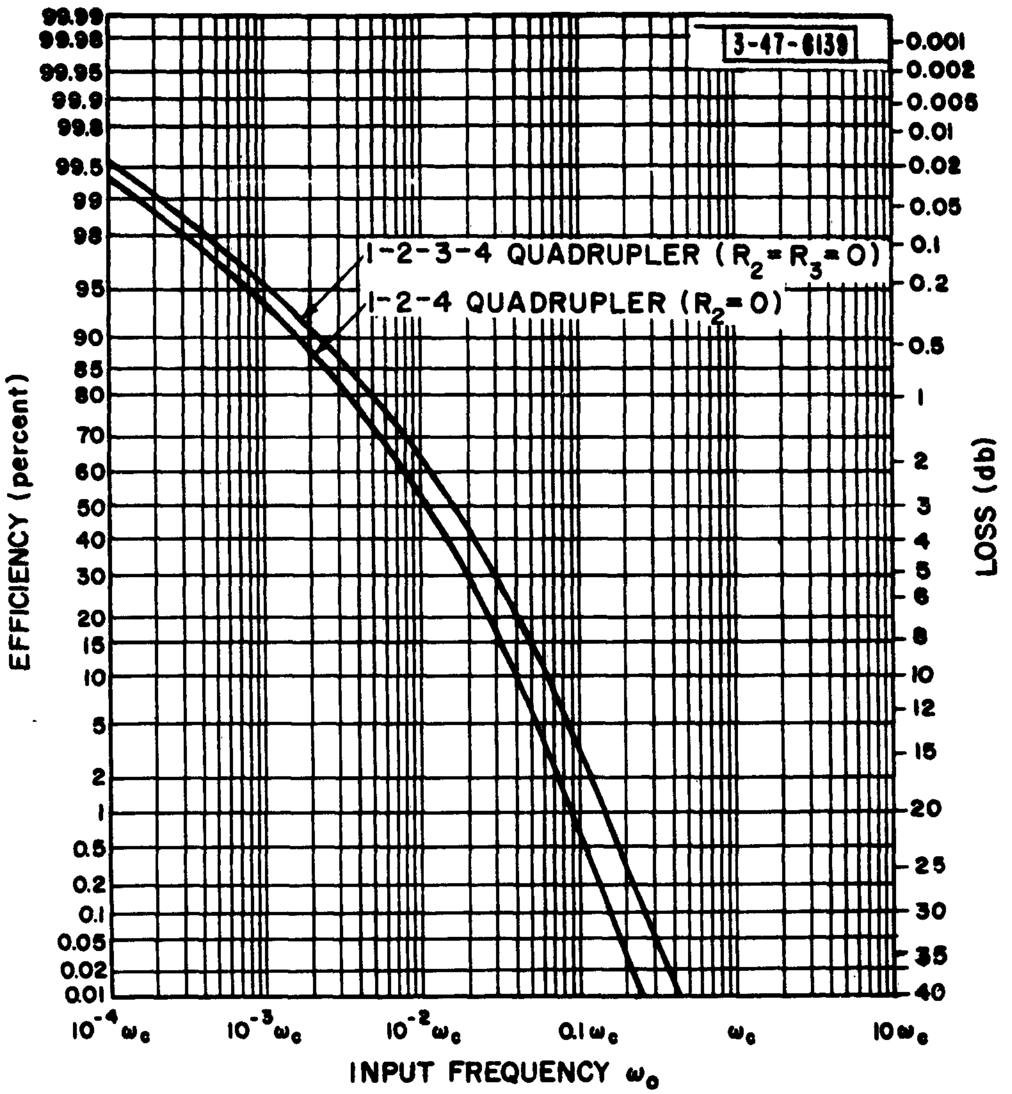

Fig. 4.17 Comparison of the maximum efficiency obtainable from the abrupt-junction-varactor 1-2-4 and 1-2-3-4 quadruplers. In both cases the idler terminations are lossless.

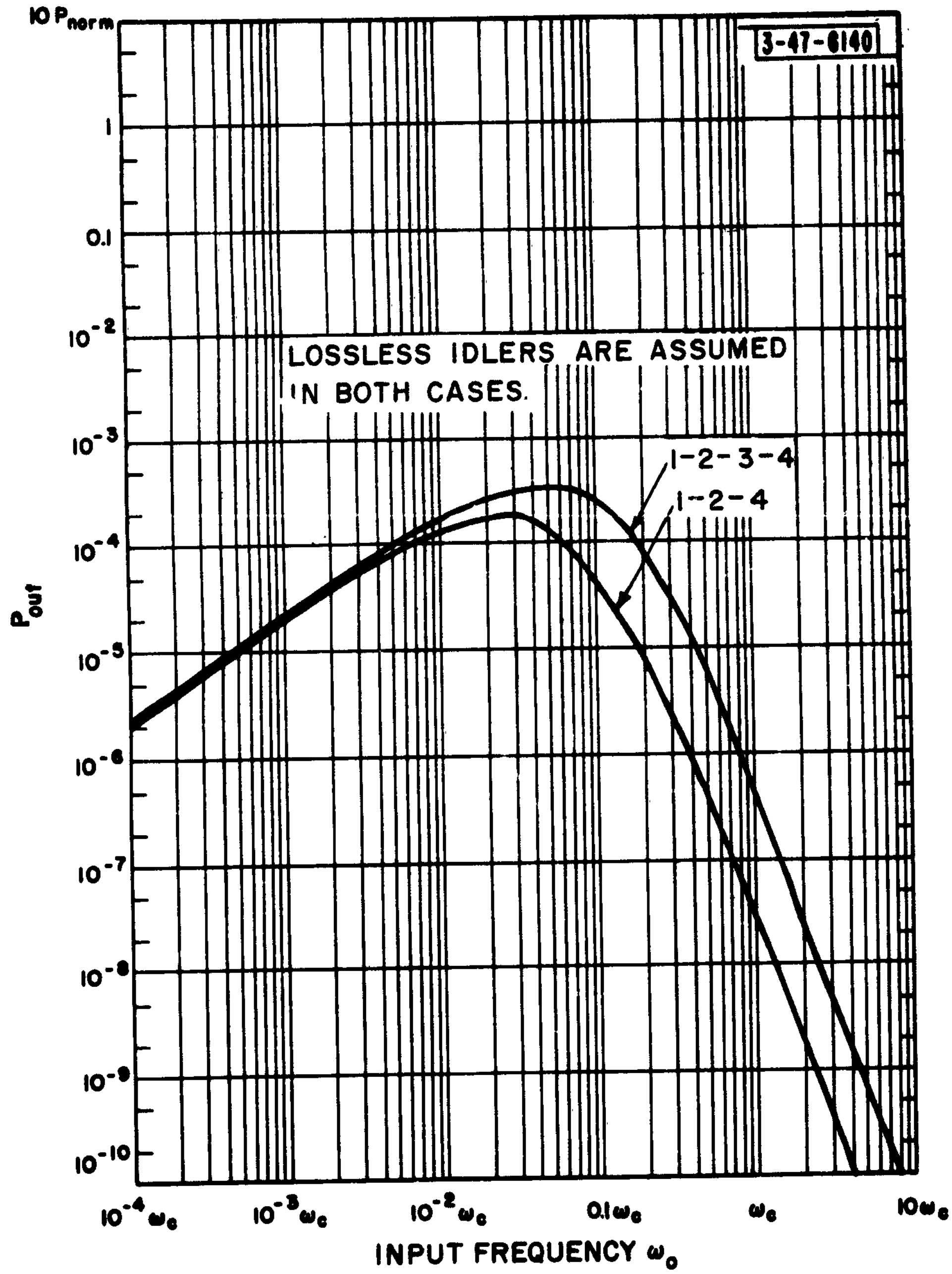

Fig. 4.18 Comparison of the power output for maximum efficiency operation with lossless idler terminations for the abrupt-junction-varactor 1-2-4 and 1-2-3-4 quadruplers.

V. QUINTUPLER SOLUTIONS

Unlike the tripler and the 1-2-4 quadrupler, quintuplers require two or more idlers. There are two possible idler configurations for the abrupt-junction-varactor quintupler with only two idlers. The first has idlers at the second and fourth harmonics (1-2-4-5), and the other at the second and third harmonics (1-2-3-5). Many other quintuplers are conceivable, but all of them have three or more idlers. We will restrict our attention to the two-idler quintuplers, even though some multiple-idler device may actually yield better performance. There appears to be no a priori way of determining which of the two-idler quintuplers is better so both must be analyzed. Detailed investigation of the 1-2-3-5 quintupler formulas reveals an unusual behavior which probably causes this device to have no practical value except at high frequencies. No direct comparison will be made because of this anomaly.

5.1 1-2-4-5 Quintupler Formulas

From Eq. (2.24) the idler and load resistance equations are

$$\frac{R_5 + R_s}{R_s} = \frac{\omega_c}{5\omega_o} \frac{(jM_1)(jM_4)}{jM_5} , \tag{5.1}$$

$$\frac{R_4 + R_s}{R_s} = \frac{\omega_c}{8\omega_o} \frac{(jM_2)^2 - 2(jM_1)^*(jM_5)}{jM_4} , \tag{5.2}$$

$$\frac{R_2 + R_s}{R_s} = \frac{\omega_c}{4\omega_o} \frac{(jM_1)^2 - 2(jM_2)^*(jM_4)}{jM_2} . \tag{5.3}$$

The time origin may be chosen such that jM_1 is real and positive and thus equal to its magnitude, m_1. Equation (5.1) then shows that the phase angle of jM_5 is equal to the phase angle of jM_4, and Eq. (5.2) indicates that the phase angle of jM_4 is twice the phase angle of jM_2. Using this information

in Eq. (5.3), we see that the phase angle of jM_2 must be zero, so that $jM_1 = m_1$, $jM_2 = m_2$, $jM_4 = m_4$, and $jM_5 = m_5$.

In terms of the m_k the various 1-2-4-5 quintupler formulas are

$$R_5 = R_s\left(\frac{\omega_c}{5\omega_o}\frac{m_1 m_4}{m_5} - 1\right) , \tag{5.4}$$

$$R_4 = R_s\left(\frac{\omega_c}{8\omega_o}\frac{m_2^2 - 2m_1 m_5}{m_4} - 1\right) , \tag{5.5}$$

$$R_2 = R_s\left(\frac{\omega_c}{4\omega_o}\frac{m_1^2 - 2m_2 m_4}{m_2} - 1\right) , \tag{5.6}$$

$$R_{in} = R_s\left(\frac{\omega_c}{\omega_o}\frac{m_1 m_2 + m_4 m_5}{m_1} + 1\right) , \tag{5.7}$$

$$\frac{P_{in}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 \left[\frac{\omega_c}{\omega_o} m_1(m_1 m_2 + m_4 m_5) + m_1^2\right] , \tag{5.8}$$

$$\frac{P_{out}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 \left[\frac{5\omega_c}{\omega_o} m_1 m_4 m_5 - 25m_5^2\right] , \tag{5.9}$$

$$\frac{P_{diss}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 \left[\frac{\omega_c}{\omega_o} m_1(m_1 m_2 - 4m_4 m_5) + m_1^2 + 25m_5^2\right] , \tag{5.10}$$

$$\frac{P_{diss,v}}{P_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 (m_1^2 + 4m_2^2 + 16m_4^2 + 25m_5^2) , \tag{5.11}$$

$$\epsilon = \frac{5m_5}{m_1}\,\frac{\left(\frac{\omega_c}{\omega_o}\right) m_1 m_4 - 5m_5}{\left(\frac{\omega_c}{\omega_o}\right)(m_1 m_2 + m_4 m_5) + m_1} , \tag{5.12}$$

$$\frac{V_o + \varphi}{V_B + \varphi} = \frac{(S_{max} - S_{min})^2}{S_{max}^2} \left[\frac{1}{4} + 2(m_1^2 + m_2^2 + m_4^2 + m_5^2)\right] + \frac{S_{min}}{S_{max}} , \tag{5.13}$$

where we have set $m_o = 1/2$. The power relations have been normalized with respect to P'_{norm} as defined by Eq. (3.12).

The M_k have been shown to be entirely imaginary, so Condition (2.17) can be used as the bound on the magnitudes of the m_k. For this case we have

$$m_1 \sin \omega_o t + m_2 \sin 2\omega_o t + m_4 \sin \omega_o t + m_5 \sin 5\omega_o t \leq 0.25 \tag{5.14}$$

for all values of t.

5.2 Solution of the 1-2-4-5 Quintupler Equations

The solution of these equations is similar to the solution of the other abrupt-junction-varactor multiplier equations. If, for example, we know values of m_1, m_2, m_4, and m_5 which satisfy Condition (5.14) and which give positive values for R_2, R_4, and R_5, then all quantities of interest can be calculated.

As discussed in connection with the other multipliers, we are usually faced with the inverse problem of finding the loading conditions which maximize either the efficiency or the power output. Both of these maxima occur when the elastance attains the values of S_{min} and S_{max} one or more times during each cycle, that is, when Condition (5.14) is satisfied with the equality sign at the time t_o when $m(t)$ is a maximum or a minimum.

The tripler and quadrupler equations were solved by an iterative numerical procedure. The same is true for the 1-2-4-5 quintupler. As with the other multipliers, we usually choose values for R_2, R_4, R_5, and ω_o/ω_c and then compute the required values for the m_k by iteration. It is therefore convenient to solve Eqs. (5.4), (5.5), and (5.6) for three of the m_k in terms of the other one. The equations are found to be simplest if m_2 is used as the reference. Thus,

$$\left(\frac{m_4}{m_2}\right)^2 \left(\frac{4}{5} \frac{R_s}{R_5 + R_s} \frac{m_2 \omega_c}{\omega_o}\right)$$

$$+ \frac{m_4}{m_2} \left(\frac{8}{5} \frac{R_2 + R_s}{R_5 + R_s} + \frac{R_4 + R_s}{R_s} \frac{8\omega_o}{m_2 \omega_c}\right) - 1 = 0, \tag{5.15}$$

$$\left(\frac{m_1}{m_2}\right)^2 = 2 \frac{m_4}{m_2} + \frac{R_2 + R_s}{R_s} \frac{4\omega_o}{m_2 \omega_c} , \tag{5.16}$$

$$\frac{m_5}{m_2} = \frac{R_s}{R_5 + R_s} \frac{m_2 \omega_c}{5\omega_o} \frac{m_1}{m_2} \frac{m_4}{m_2} . \tag{5.17}$$

The numerical procedure for solving these equations is the same as described in Section 3. 2 for the tripler, except that in this case m_2 is the control parameter. In performing the calculations, it is important that we take only the positive real root of Eq. (5. 15), since it is the only one which satisfies the phase condition. The pertinent computations for maximum efficiency operation and maximum power output operation have been performed on an I. B. M. 7090 digital computer. Again we find that the efficiency, power input, and power output as functions of R_5 for a given frequency and specific values of R_2 and R_4 look somewhat like Fig. 3. 1 for the tripler. For small values of the idler resistances we also find that, for practical purposes, efficiency and power output are simultaneously maximized.

The computed results for maximum efficiency operation are presented graphically in Figs. 5. 1 to 5. 7. In these figures we show efficiency, input resistance, load resistance, power input, power output, dissipated power, and bias voltage as functions of frequency for several values of R_2 and R_4. These plots are similar to those given for the tripler and quadruplers and are interpreted in the same way.

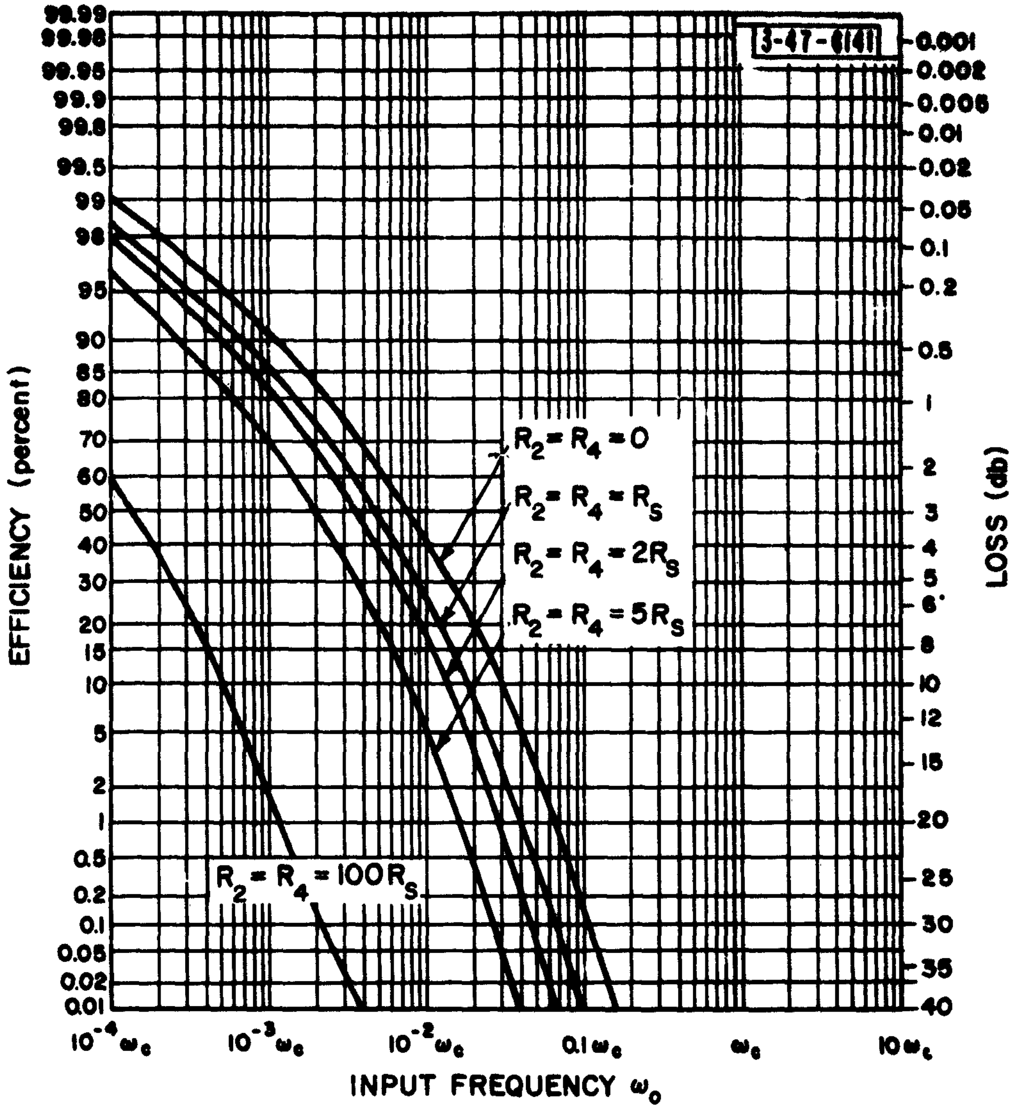

Fig. 5.1 Maximum efficiency of an abrupt-junction-varactor 1-2-4-5 quintupler for several values of idler resistances, R_2 and R_4.

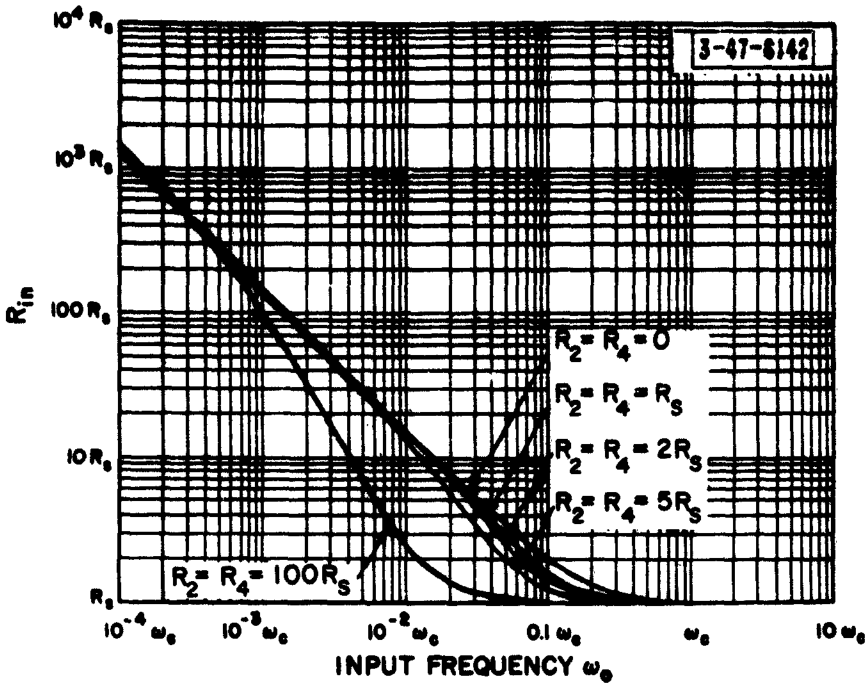

Fig. 5.2 Input resistance for a 1-2-4-5 quintupler adjusted for maximum efficiency operation.

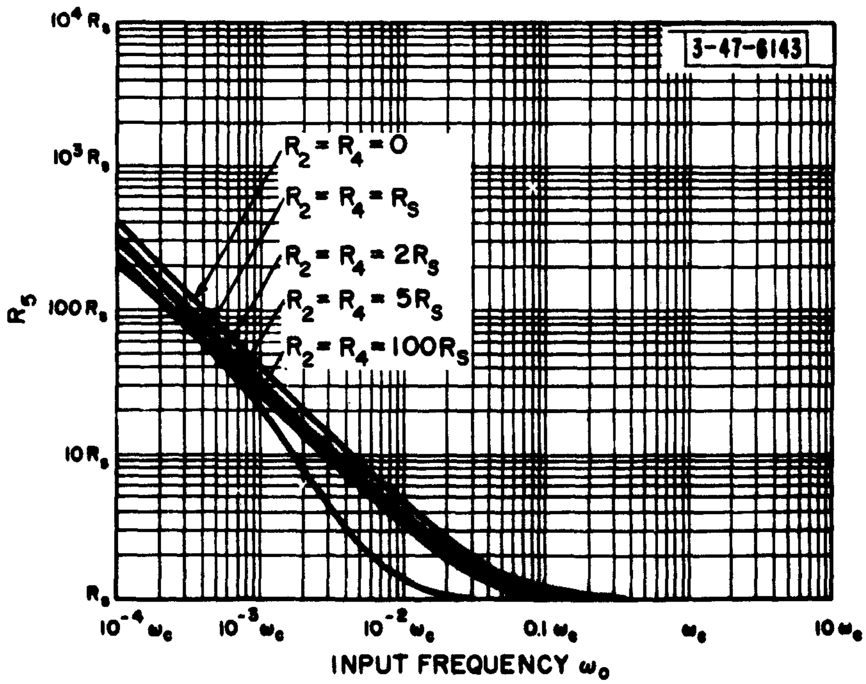

Fig. 5.3 Load resistance for maximum efficiency operation of a 1-2-4-5 quintupler for a variety of idler resistances, R_2 and R_4.

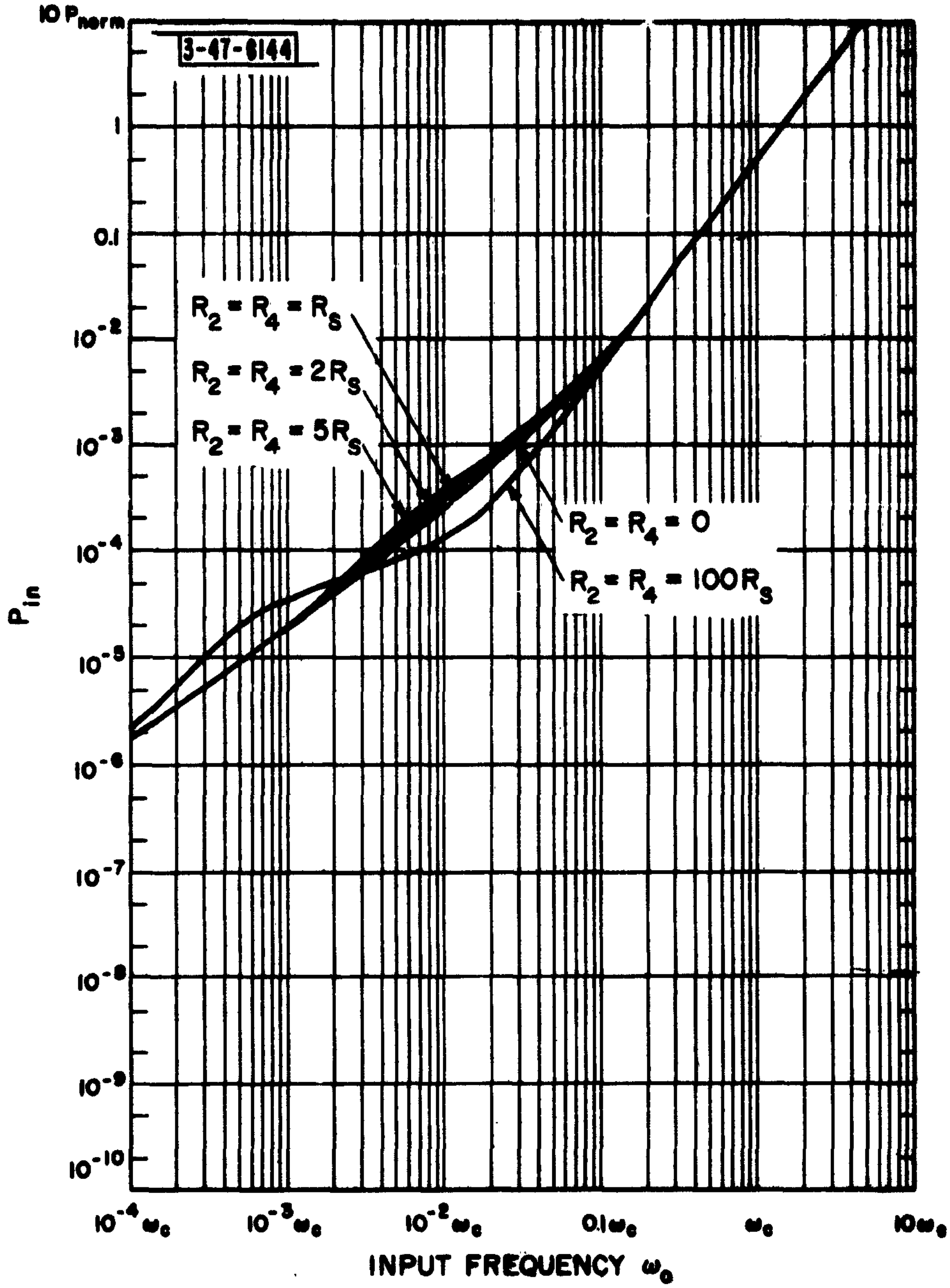

Fig. 5.4 Power input of a 1-2-4-5 quintupler adjusted for maximum efficiency operation.

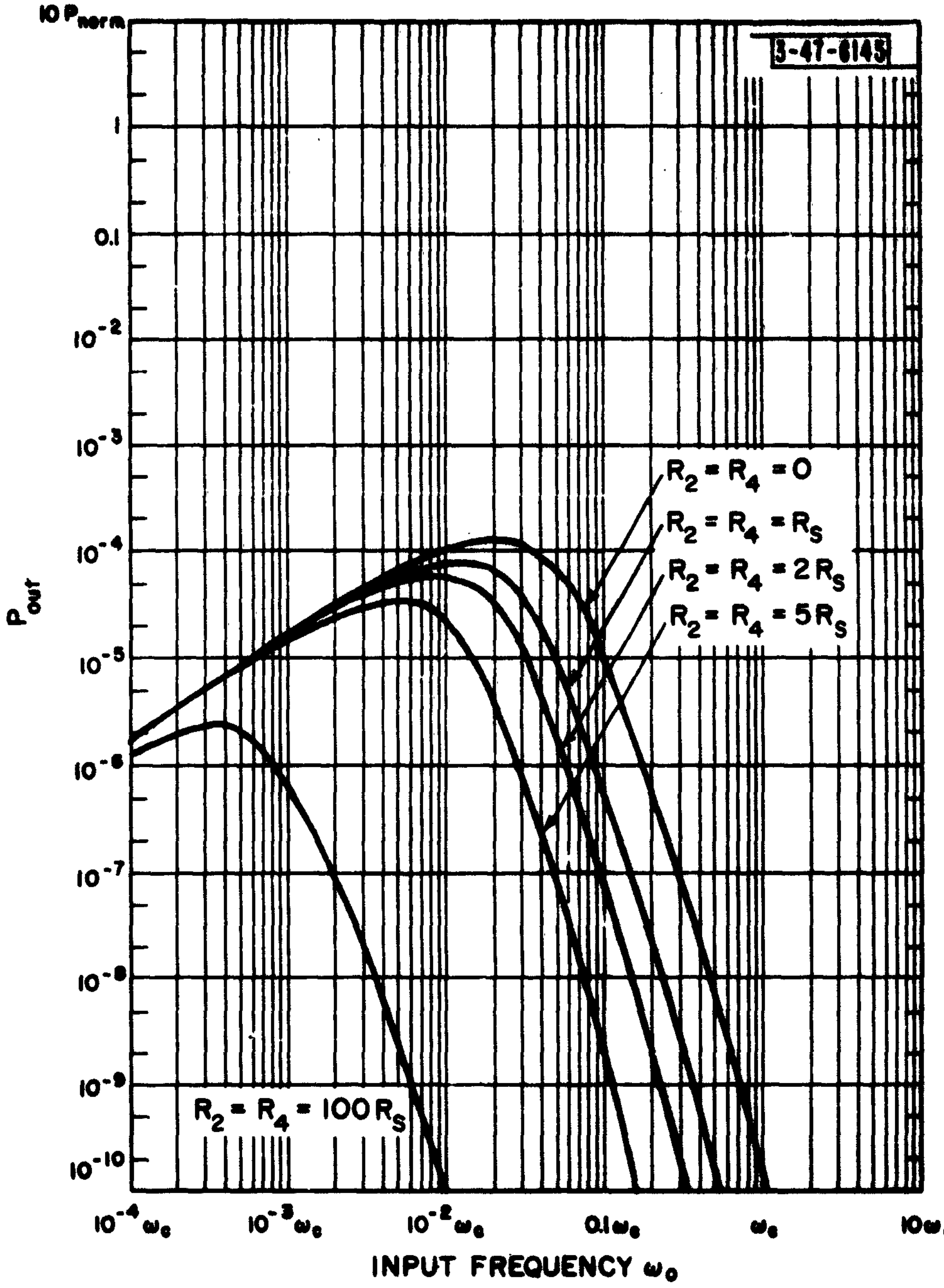

Fig. 5.5 Power output for maximum efficiency operation of a 1-2-4-5 quintupler for several values of the idler resistances, R_2 and R_4.

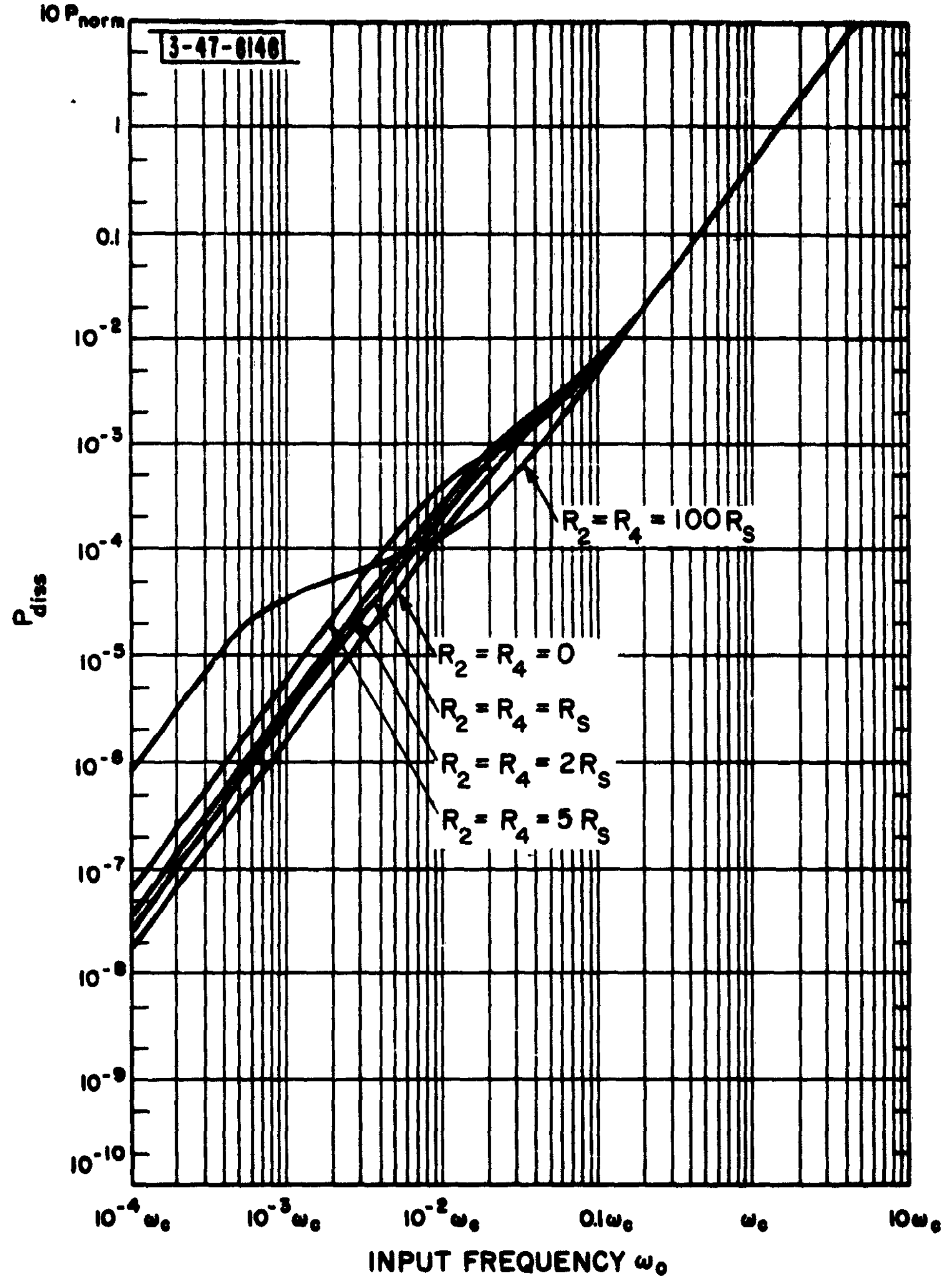

Fig. 5.6 Total power dissipated in a 1-2-4-5 quintupler adjusted for maximum efficiency operation.

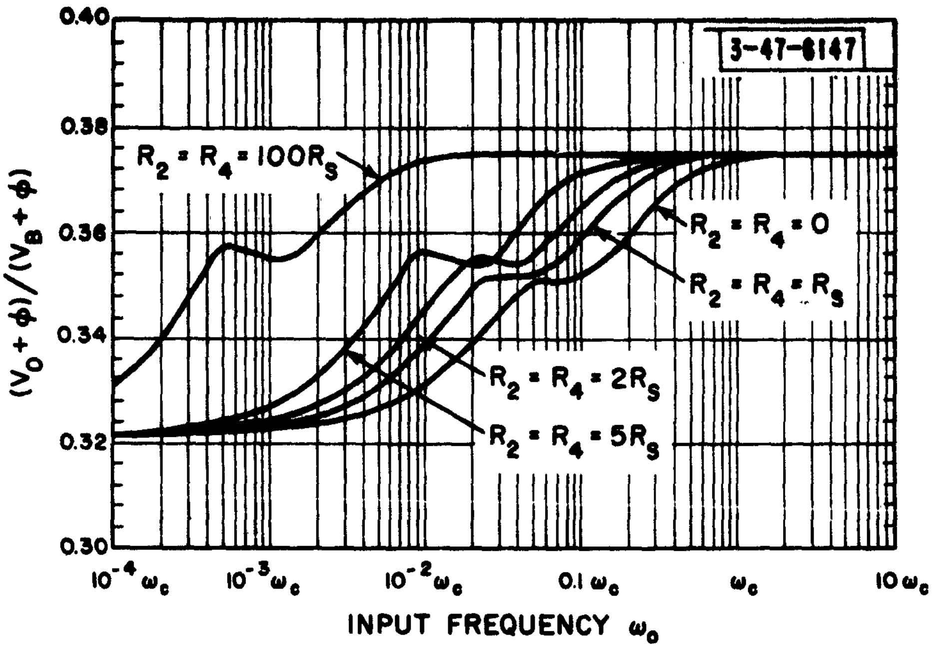

Fig. 5.7 Bias voltage for a 1-2-4-5 quintupler adjusted for maximum efficiency operation for a variety of idler resistances, R_2 and R_4.

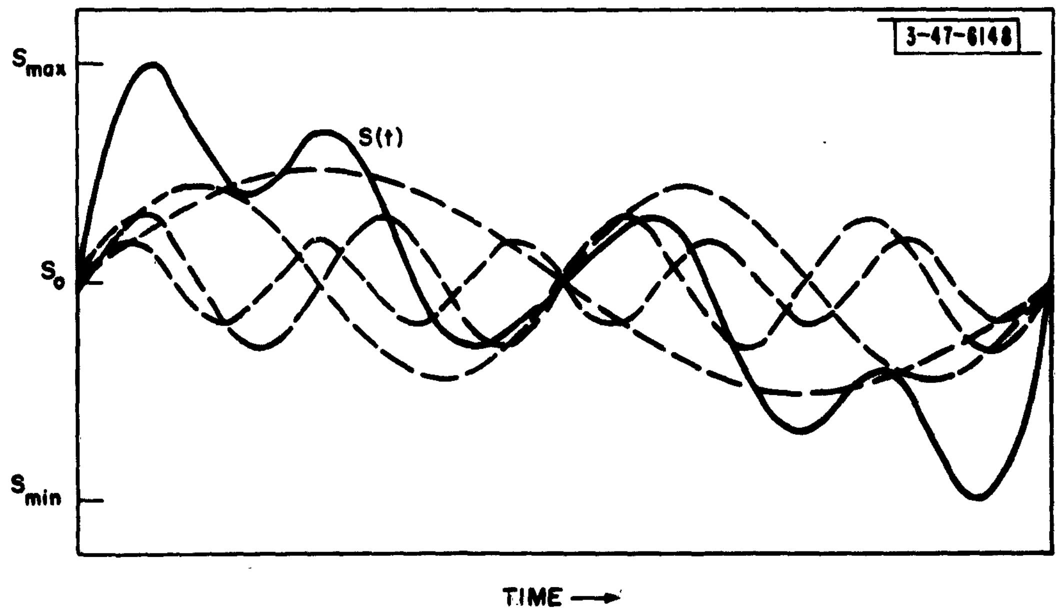

Fig. 5.8 Elastance waveform of a low-frequency 1-2-4-5 quintupler adjusted for maximum efficiency operation with lossless idler terminations.

The minimum elastance, S_{min}, has been neglected in Figs. 5.4 to 5.7. If S_{min} is non-zero, we must use P'_{norm} instead of P_{norm} for the normalization power, and the bias voltage as given in Fig. 5.7 must be modified according to Eq. (5.13). The average elastance, S_o, is given by Eq. (3.19), since our solutions are for maximum drive.

The conditions for maximum power output are very similar to those for maximum efficiency, at least for small values of R_2 and R_4. For large values of the idler resistances the optimizations are quite different, but we do not plot the results because in practice large idler resistances are generally undesirable.

A plot of the elastance waveform is shown in Fig. 5.8 for maximum efficiency operation at low frequency with lossless idlers. For maximum drive (as shown in the figure) S_{min} and S_{max} are attained at least once per cycle. With some loading conditions, it is possible to attain S_{min} and S_{max} twice per cycle. Care must, therefore, be taken in performing the calculations to make sure that the higher maximum is used in checking Condition (5.14).

5.3 Asymptotic Formulas for the 1-2-4-5 Quintupler

At low and high frequencies the behavior of this multiplier can be described by asymptotic formulas. The limiting values of the m_k for low frequencies, as found from the computed data, are used in Eqs. (5.4) to (5.13) to determine the appropriate formulas. For high frequencies we use the limiting values, $R_{in} \approx R_5 \approx R_s$, $m_1 \approx 0.25$, and $m_k \ll m_1$ for $k > 1$, in Eqs. (5.4) to (5.13) to find the asymptotic relations. These formulas are summarized in Table 5.1 for the case of lossless idlers. The asymptotic relations for low frequencies are almost identical for both maximum efficiency and maximum power output operation. Therefore, only one set of low-frequency formulas is given.

5.4 Formulas for the 1-2-3-5 Quintupler

The formulas developed in Chapter II apply to this case with all M_k zero except for $k = 0$, ± 1, ± 2, ± 3, and ± 5. From Eq. (2.24) the idler and load resistance equations are

Table 5.1 Asymptotic Formulas for the 1-2-4-5 Quintupler

Maximum power output and maximum efficiency are achieved with lossless idlers, so we have set $R_2 = R_4 = 0$. For simplicity we have also assumed $S_{min} \ll S_{max}$.

	Low frequency Maximum ϵ and P_{out}	High frequency Maximum ϵ and P_{out}
ϵ	$1 - 92.9(\frac{\omega_o}{\omega_c})$	$2.33 \times 10^{-10}(\frac{\omega_c}{\omega_o})^8$
R_{in}	$0.136(\frac{\omega_c}{\omega_o}) R_s$	R_s
R_5	$0.209(\frac{\omega_c}{5\omega_o}) R_s$	R_s
$\frac{P_{in}}{P_{norm}}$	$0.0178(\frac{\omega_o}{\omega_c})$	$0.500(\frac{\omega_o}{\omega_c})^2$
$\frac{P_{out}}{P_{norm}}$	$0.0178(\frac{\omega_o}{\omega_c})$	$1.16 \times 10^{-10}(\frac{\omega_c}{\omega_o})^6$
$\frac{P_{diss}}{P_{norm}}$	$1.65(\frac{\omega_o}{\omega_c})^2$	$0.500(\frac{\omega_o}{\omega_c})^2$
$\frac{V_o + \varphi}{V_B + \varphi}$	0.322	0.375
m_1	0.128	0.250
m_2	0.109	$0.0156(\frac{\omega_c}{\omega_o})$
m_4	0.075	$3.05 \times 10^{-5}(\frac{\omega_c}{\omega_o})^3$
m_5	0.046	$7.63 \times 10^{-7}(\frac{\omega_c}{\omega_o})^4$

$$\frac{R_5 + R_s}{R_s} = \frac{\omega_c}{5\omega_o} \frac{(jM_2)(jM_3)}{jM_5} \quad , \tag{5.18}$$

$$\frac{R_3 + R_s}{R_s} = \frac{\omega_c}{3\omega_o} \frac{(jM_1)(jM_2) - (jM_2)^*(jM_5)}{jM_3} \quad , \tag{5.19}$$

$$\frac{R_2 + R_s}{R_s} = \frac{\omega_c}{4\omega_o} \frac{(jM_1)^2 - 2(jM_1)^*(jM_3) - 2(jM_3)^*(jM_5)}{jM_2} \quad . \tag{5.20}$$

Without loss of generality, we may choose the time origin such that jM_1 is real and positive and thus equal to its magnitude, m_1. Equation (5.18) shows that the phase angle of jM_5 equals the sum of the phase angles of jM_2 and jM_3. When this information is inserted in Eq. (5.19), we find that the phase angle of jM_3 equals the phase angle of jM_2. Finally, from Eq. (5.20) we find that jM_2 must be real and positive. It then follows that jM_3 and jM_5 are real and positive and equal to their magnitudes, m_3 and m_5, respectively.

In terms of the m_k the various 1-2-3-5 quintupler formulas are

$$R_5 = R_s \left[\frac{\omega_c}{5\omega_o} \frac{m_2 m_3}{m_5} - 1 \right] \quad , \tag{5.21}$$

$$R_3 = R_s \left[\frac{\omega_c}{3\omega_o} \frac{m_1 m_2 - m_2 m_5}{m_3} - 1 \right] \quad , \tag{5.22}$$

$$R_2 = R_s \left[\frac{\omega_c}{4\omega_o} \frac{m_1^2 - 2m_1 m_3 - 2m_3 m_5}{m_2} - 1 \right] \quad , \tag{5.23}$$

$$R_{in} = R_s \left[\frac{\omega_c}{\omega_o} \frac{m_1 m_2 + m_2 m_3}{m_1} + 1 \right] \quad , \tag{5.24}$$

$$\frac{P_{in}}{P_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 \left[\frac{\omega_o}{\omega_c} m_1 m_2 (m_1 + m_3) + m_1^2 \right] \quad , \tag{5.25}$$

$$\frac{P_{out}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 \left[\frac{5\omega_c}{\omega_o} m_2 m_3 m_5 - 25m_5^2\right] \quad , \tag{5.26}$$

$$\frac{P_{diss}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 \left[\frac{\omega_c}{\omega_o} m_2(m_1^2 + m_1 m_3 - 5m_3 m_5) + m_1^2 + 25m_5^2\right] , \tag{5.27}$$

$$\frac{P_{diss,v}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 (m_1^2 + 4m_2^2 + 9m_3^2 + 25m_5^2) \quad , \tag{5.28}$$

$$\epsilon = \frac{5m_5}{m_1} \frac{\frac{\omega_c}{\omega_o} m_2 m_3 - 5m_5}{\frac{\omega_c}{\omega_o} m_2(m_1 + m_3) + m_1} \quad , \tag{5.29}$$

$$\frac{V_o + \varphi}{V_B + \varphi} = \left(\frac{S_{max} - S_{min}}{S_{max}}\right)^2 \left[\frac{1}{4} + 2(m_1^2 + m_2^2 + m_3^2 + m_5^2)\right] + \frac{S_{min}}{S_{max}} \quad , \tag{5.30}$$

where we have set $m_o = 1/2$. The power relations have been normalized with respect to P'_{norm} as defined by Eq. (3.12).

The M_k have been shown to be entirely imaginary, so Condition (2.17) can be used as the bound on the magnitudes of the m_k. For this case we have

$$m_1 \sin \omega_o t + m_2 \sin 2\omega_o t + m_3 \sin 3\omega_o t + m_5 \sin 5\omega_o t \le 0.25 \tag{5.31}$$

for all values of t.

5.5 Technique of Solution of the 1-2-3-5 Quintupler

The solution of these equations is similar to the solution of the other abrupt-junction-varactor multiplier equations. If, for example, we know values for the m_k which satisfy Condition (5.31) and which give positive values for R_2, R_3, and R_5, then all quantities of interest can be calculated.

More often we must find values for the m_k which are compatible with prescribed values of the resistances. In this case it is convenient to have Eqs. (5.21) to (5.23) solved for the ratios of the m_k. Any one of the m_k could be chosen as the reference, but the choice of m_2 leads to the simplest formulas:

$$\left(\frac{m_3}{m_2}\right)^2 = \frac{\dfrac{4\omega_o}{m_2\omega_c}\dfrac{R_2+R_s}{R_s}}{\left(\dfrac{3\omega_o}{m_2\omega_c}\dfrac{R_3+R_s}{R_s}+\dfrac{m_2\omega_c}{5\omega_o}\dfrac{R_s}{R_5+R_s}\right)^2-\dfrac{6\omega_o}{m_2\omega_c}\dfrac{R_3+R_s}{R_s}-\dfrac{4}{5}\dfrac{m_2\omega_c}{\omega_o}\dfrac{R_s}{R_5+R_s}}, \tag{5.32}$$

$$\frac{m_1}{m_2} = \left(\frac{3\omega_o}{m_2\omega_c}\frac{R_3+R_s}{R_s}+\frac{m_2\omega_c}{5\omega_o}\frac{R_s}{R_5+R_s}\right)\frac{m_3}{m_2}\quad, \tag{5.33}$$

$$\frac{m_5}{m_2} = \left(\frac{m_2\omega_c}{5\omega_o}\frac{R_s}{R_5+R_s}\right)\frac{m_3}{m_2}\quad. \tag{5.34}$$

The numerical procedure for solving this multiplier is much the same as for the other multipliers. However, this quintupler exhibits an unusual behavior at low to moderate frequencies which probably makes it impractical.

5.6 Anomalous Behavior of the 1-2-3-5 Quintupler

The 1-2-3-5 quintupler has an unusual property that can be appreciated by studying Eq. (5.32). We recall that the phase condition requires the jM_k to be real and positive. Therefore, the right hand side of Eq. (5.32) must be positive. In particular, the denominator of this equation must be positive, which places a restriction on the permissible values of ω_o/ω_c, m_2, R_3, and R_5. This condition is separate from the requirement that the elastance vary between S_{min} and S_{max}.

Further information about this restriction can be obtained by noting that the denominator of Eq. (5.32) is a function of only two quantities:*

$$\frac{m_2 \omega_c}{3\omega_o} \frac{R_s}{R_3 + R_s} \tag{5.35}$$

and

$$\frac{m_2 \omega_c}{5\omega_o} \frac{R_s}{R_5 + R_s} \quad . \tag{5.36}$$

Figure 5.9 shows clearly the combinations of these quantities for which the denominator is positive (or negative). The solid curve is for the values which make the denominator zero.

As m_2 is varied for specific values of R_3 and R_5, both the abscissa and the ordinate of Fig. 5.9 are increased simultaneously along a straight line passing through the origin. The dashed line in Fig. 5.9 shows one such line for $R_3 = R_5 \geq 0$. If m_2 is small, then operation is in an allowed region near the origin. If m_2 is high enough while Condition (5.31) is still satsified, operation is again permissible. Between these two regions there is a forbidden region which raises the important question as to whether operation can pass from one allowed region to the other.

Consider a quintupler with values of R_3 and R_5 such that the operating line passes through the forbidden region, for example, the dashed line in Fig. 5.9. (Most practical multipliers would have operating lines that pass through the forbidden region, since R_3 is usually small while R_5 is large.) Before the multiplier is excited, $m_2 = 0$ and the point describing the operation is located at the origin. If the input power is gradually increased, m_2 increases, and the operation is described in Fig. 5.9 by a point that moves along the line indicated. For some value of the input power, the operating point will encounter the boundary of the forbidden region from the left in Fig. 5.9 (assuming, of course, that Condition (5.14) is not violated before

*This method of investigating the anomaly was pointed out to the author by P. Penfield, Jr.

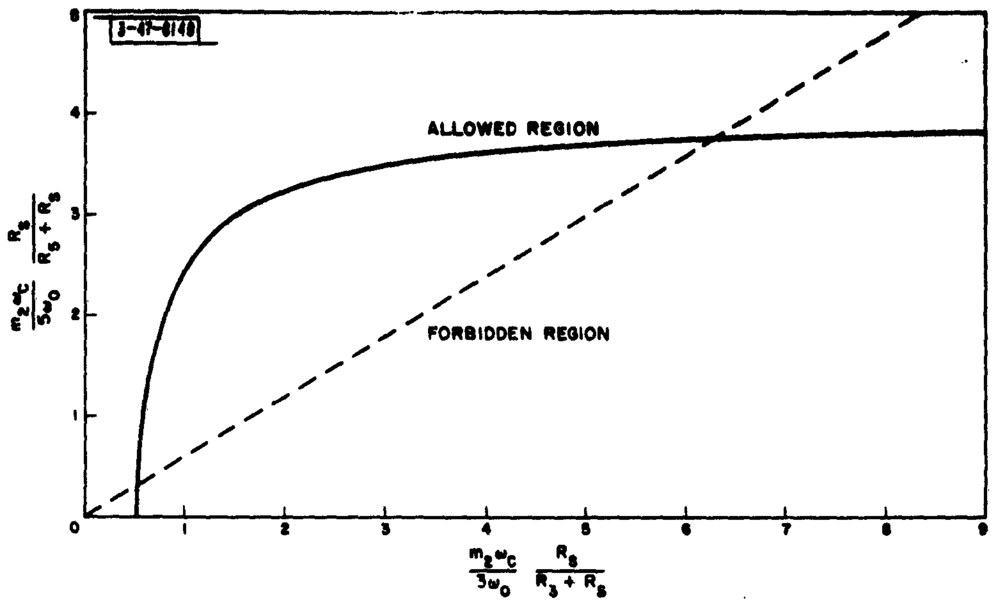

Fig. 5.9 Assumed values of R_3, R_5, ω_o/ω_c, and m_2 must be compatible with the figure for the calculated values of m_1, m_3, and m_5 to be real. Lines are formed on the plot by varying m_2 for fixed values of R_3 and R_5. The dashed line is for $R_3 = R_5$. The region below the line is for $R_5 > R_3$, and the region above the line is for $R_5 < R_3$.

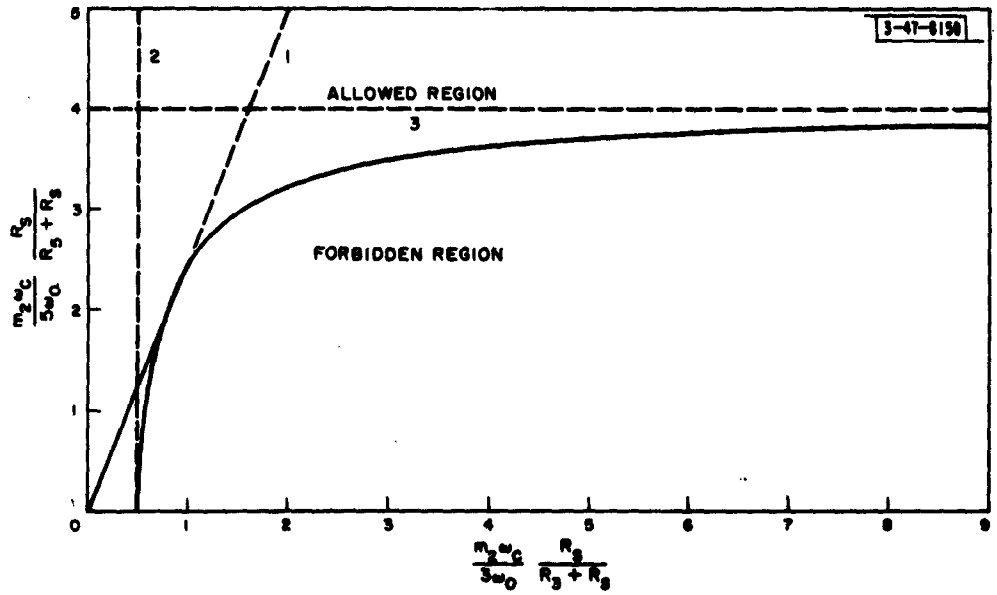

Fig. 5.10 The lines shown on the plot give bounds on the values of m_2, R_3, R_5, and ω_o/ω_c such that the 1-2-3-5 quintupler will always be operating in an allowed region. The lines are plots of Conditions (5.37), (5.38), and (5.39) with equality signs.

this point is reached). The operating point cannot move into the forbidden region because the phase condition would be violated, and it cannot depart from the operating line because R_3 and R_5 are fixed. Therefore, the operating point remains near the boundary of the forbidden region as the power input is further increased. Finally, the power input reaches a point where the varactor is fully driven, but the operating point has not moved any significant amount. Any further increase in the input power drives the varactor into either conduction or avalanche breakdown (or both). When this occurs the resulting nonlinear effects (or, perhaps, transient effects) may cause the multiplier operation to switch to the allowed region beyond the forbidden region. It is not clear that this will happen and even if it does it may be an undesirable feature for a practical multiplier.

The above argument indicates that the 1-2-3-5 quintupler will usually be operating in a region very near the origin in Fig. 5.9 unless some transient effect occurs such that operation is switched to another segment of the allowed region where m_2 is large. Operation near the origin corresponds to small values of m_2 for which the efficiency is low. Conversely, large values of m_2 give high efficiency. This leads us to conclude that this multiplier would normally be operating in a low-efficiency mode or, perhaps, it might be susceptible to "jumping" from high- to low-efficiency operation (or vice versa) due to power level changes. Neither of these possibilities are desirable in a practical multiplier. Therefore, we are inclined to reject the 1-2-3-5 quintupler as a low-frequency multiplier.

There are certain regions of operation for this multiplier in which the forbidden region is never encountered. For example, if the operating line is steep enough, then operation is always in the allowed region. In particular, if

$$\frac{R_3 + R_s}{R_5 + R_s} \geq 4.17 \quad , \tag{5.37}$$

then all values of m_2 are allowed. This corresponds to the region to the left of line (1) in Fig. 5.10. If Condition (5.37) does not hold, then some

values of m_2 are forbidden (for example, if $R_3 < 3.17\ R_s$, some values of m_2 are forbidden because R_5 must be non-negative).

Another bound which assures that we will be operating in the allowed region is

$$\frac{m_2 \omega_c}{\omega_o} \frac{R_s}{R_3 + R_s} \leq 1.5 \quad . \tag{5.38}$$

This corresponds to the region to the left of line (2) in Fig. 5.10. Condition (5.38) is always satisfied for high frequencies so the 1-2-3-5 quintupler can be used as a high-frequency multiplier. Similarly, we are above the forbidden region (above line (3) in Fig. 5.10), if

$$\frac{m_2 \omega_c}{\omega_o} \frac{R_s}{R_5 + R_s} \geq 20.0 \quad . \tag{5.39}$$

If any one or more of Conditions (5.37) to (5.39) holds, the corresponding values of ω_o/ω_c, m_2, R_3, and R_5 are allowable. It should be emphasized that this condition is separate from Condition (5.31); values that are compatible with Fig. 5.9 are not necessarily realizable. Both Condition (5.31) and the requirements of Fig. 5.9 must hold.

5.7 High-Frequency Solution of the 1-2-3-5 Quintupler

At high frequencies Condition (5.38) always holds. For these frequencies, the 1-2-3-5 quintupler exhibits an efficiency several times that of the 1-2-4-5 quintupler. The 1-2-3-5 quintupler also has a power output greater than that of the 1-2-4-5 quintupler. The pertinent high-frequency formulas are summarized in Table 5.2. In deriving the formulas given in this table, we have used the limiting values, $R_{in} \approx R_5 \approx R_s$, $m_1 \approx 0.25$, and $m_k \ll m_1$ for $k > 1$, in Eqs. (5.21) to (5.30). We have also assumed that the idler terminations are lossless, that the varactor is fully pumped, that S_{min} is negligible, and that the output load is adjusted for maximum power output and efficiency.

Table 5.2 Asymptotic Formulas for the 1-2-3-5 Quintupler

High-frequency behavior of the abrupt-junction-varactor 1-2-3-5 quintupler with lossless idler terminations. We have neglected S_{min} in comparison to S_{max}.

$\epsilon \approx 16.6 \times 10^{-10}\left(\frac{\omega_c}{\omega_o}\right)^8$	$P_{in} \approx 0.500\ P_{norm}\left(\frac{\omega_o}{\omega_c}\right)^2$
$R_{in} \approx R_s$	$P_{out} \approx 8.28 \times 10^{-10}\ P_{norm}\left(\frac{\omega_c}{\omega_o}\right)^2$
$R_5 \approx R_s$	$P_{diss} \approx 0.500\ P_{norm}\left(\frac{\omega_o}{\omega_c}\right)^2$
$\frac{V_o + \varphi}{V_B + \varphi} \approx 0.375$	
$m_1 \approx 0.250$	$m_3 \approx 0.0013\left(\frac{\omega_c}{\omega_o}\right)^2$
$m_2 \approx 0.0156\ \frac{\omega_c}{\omega_o}$	$m_5 \approx 2.03 \times 10^{-6}\left(\frac{\omega_c}{\omega_o}\right)^4$

VI. SEXTUPLER SOLUTIONS

The sextupler, like the quintupler, requires two or more idler currents. Two sextuplers are possible with two idlers. One has idlers at the second and fourth harmonics (1-2-4-6), and the other has idlers at the second and third harmonics (1-2-3-6). There appears to be no a priori way of deciding which of these is better, or in fact of knowing whether either is better than a sextupler with more than two idlers. In this chapter we restrict our attention to the two-idler sextuplers. The two sextuplers will be compared to demonstrate the differences in their expected performances.

6.1 1-2-4-6 Sextupler Formulas

From Eq. (2.24) the idler and load resistance equations are

$$\frac{R_6 + R_s}{R_s} = \frac{\omega_c}{6\omega_o} \frac{(jM_2)(jM_4)}{jM_6} , \tag{6.1}$$

$$\frac{R_4 + R_s}{R_s} = \frac{\omega_c}{8\omega_o} \frac{(jM_2)^2 - 2(jM_2)^*(jM_6)}{jM_4} , \tag{6.2}$$

$$\frac{R_2 + R_s}{R_s} = \frac{\omega_c}{4\omega_o} \frac{(jM_1)^2 - 2(jM_2)^*(jM_4) - 2(jM_4)^*(jM_6)}{jM_2} . \tag{6.3}$$

We choose the time origin such that jM_1 is real and positive and thus equal to its magnitude, m_1. Equation (6.1) shows that the phase angle of jM_6 is equal to the sum of the phase angles of jM_2 and jM_4. When this information is used in Eq. (6.2), we find that the angle of jM_4 is twice the angle of jM_2. Finally, Eq. (6.3) shows that the phase angle of jM_2 is zero. Thus, $jM_2 = m_2$, $jM_4 = m_4$, and $jM_6 = m_6$, where, as usual, the m_k are the magnitudes of the M_k.

In terms of the m_k the various 1-2-4-6 sextupler formulas are

$$R_6 = R_s\left(\frac{\omega_c}{6\omega_o} \frac{m_2 m_4}{m_6} - 1\right) , \tag{6.4}$$

$$R_4 = R_s \left[\frac{\omega_c}{8\omega_o} \; \frac{m_2(m_2 - 2m_6)}{m_4} - 1 \right] , \qquad (6.5)$$

$$R_2 = R_s \left[\frac{\omega_c}{4\omega_o} \; \frac{m_1^2 - 2m_4(m_2 + m_6)}{m_2} - 1 \right] , \qquad (6.6)$$

$$R_{in} = R_s \left(\frac{\omega_c}{\omega_o} \; m_2 + 1 \right) , \qquad (6.7)$$

$$\frac{P_{in}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 m_1^2 \left(\frac{\omega_c}{\omega_o} \; m_2 + 1 \right) , \qquad (6.8)$$

$$\frac{P_{out}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 \left(\frac{6\omega_c}{\omega_o} \; m_2 m_4 m_6 - 36m_6^2 \right) , \qquad (6.9)$$

$$\frac{P_{diss}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 \left[\frac{\omega_c}{\omega_o} \; m_2(m_1^2 - 6m_4 m_6) + m_1^2 + 36m_6^2 \right] , \qquad (6.10)$$

$$\frac{P_{diss,v}}{P_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 (m_1^2 + 4m_2^2 + 16m_4^2 + 36m_6^2) , \qquad (6.11)$$

$$\epsilon = \frac{6m_6}{m_1} \; \frac{\frac{\omega_c}{\omega_o} m_2 m_4 - 6m_6}{\frac{\omega_c}{\omega_o} m_1 m_2 + m_1} , \qquad (6.12)$$

$$\frac{V_o + \varphi}{V_B + \varphi} = \left(\frac{S_{max} - S_{min}}{S_{max}} \right)^2 \left[\frac{1}{4} + 2(m_1^2 + m_2^2 + m_4^2 + m_6^2) \right] + \frac{S_{min}}{S_{max}} , \qquad (6.13)$$

where we have set $m_o = 1/2$. The power relations have been normalized with respect to P'_{norm} as defined by Eq. (3.12).

The M_k have been shown to be entirely imaginary, so Condition (2.17) can be used as the bound on the magnitudes of the m_k. For this case we have

$$m_1 \sin \omega_o t + m_2 \sin 2\omega_o t + m_4 \sin 4\omega_o t + m_6 \sin 6\omega_o t \leq 0.25 \quad (6.14)$$

for all values of t.

6.2 Solution of the 1-2-4-6 Sextupler Equations

The solution of these equations is similar to the solution of the other abrupt-junction-varactor multiplier equations. For example, all quantities of interest can be calculated, if we know values of m_1, m_2, m_4, and m_6 which satisfy Condition (6.14) and which give positive values for R_2, R_4, and R_6.

More often, however, we are interested in finding the loading conditions required for maximum efficiency or maximum power output operation. Both of these maxima occur when the elastance attains the values of S_{min} and S_{max} one or more times during each cycle, that is, when Condition (6.14) is satisfied with the equality at some time, t_o, when m(t) is a maximum.

The 1-2-4-6 sextupler equations, like those for the other abrupt-junction-varactor multipliers, are solved by an iterative numerical procedure. As with the other multipliers, we usually choose values for the idler resistances, the load resistance, and the frequency and then compute the required values for the m_k. It is, therefore, convenient to solve Eqs. (6.4), (6.5), and (6.6) for three of the m_k in terms of the other one. The equations are found to be simplest if m_2 is used as the reference:

$$\frac{m_4}{m_2} = \frac{1}{\dfrac{8\omega_o}{m_2\omega_c}\,\dfrac{R_4 + R_s}{R_s} + \dfrac{m_2\omega_c}{3\omega_o}\,\dfrac{R_s}{R_6 + R_s}} \quad , \quad (6.15)$$

$$\frac{m_6}{m_2} = \frac{m_4}{m_2}\,\frac{m_2\omega_c}{6\omega_o}\,\frac{R_s}{R_6 + R_s} \quad , \quad (6.16)$$

$$\left(\frac{m_1}{m_2}\right)^2 = \frac{4\omega_o}{m_2\omega_c} \frac{R_2 + R_s}{R_s} + 2\left(\frac{m_4}{m_2}\right)\left(1 + \frac{m_6}{m_2}\right) . \tag{6.17}$$

The numerical procedure for solving these equations is the same as described in Section 3. 2 for the tripler, except that in this case m_2 is the control parameter. Computations for maximum efficiency and maximum power output operation have been performed on an I. B. M. 7090 digital computer. Again we find that the efficiency, power output, and power input as functions of R_6 for a given frequency and specific values of R_2 and R_4 look somewhat like Fig. 3. 1 for the tripler. For small values of the idler resistances we find that, for practical purposes, efficiency and power output may be simultaneously maximized.

The computed results for maximum efficiency operation are presented in Figs. 6. 1 to 6. 7. In these figures we show efficiency, input resistance, load resistance, power input, power output, dissipated power, and bias voltage as functions of frequency for several values of R_2 and R_4. These plots are similar to those given for the other multipliers and are interpreted in the same way.

The minimum elastance S_{min} has been neglected in Figs. 6. 4 to 6. 7. If S_{min} is not negligible, we must use P'_{norm} instead of P_{norm} for the normalization power, and the bias voltage as given in Fig. 6. 7 must be modified according to Eq. (6. 13). The average elastance, S_o, is given by Eq. (3. 19), since our solutions are for maximum drive.

The conditions for maximum power output are very similar to those for maximum efficiency, at least for small values of R_2 and R_4. For large idler resistances the optimizations are quite different, but we do not plot the results because they are of little practical interest.

A plot of the elastance waveform is shown in Fig. 6. 8 for maximum efficiency operation at low frequency with lossless idlers. For maximum drive (as shown in the figure) S_{min} and S_{max} are attained at least once per cycle. With some loading conditions, it is possible to attain S_{min} and S_{max} twice per cycle. Care must, therefore, be taken in performing the calculations to make sure that the highest maximum is used in checking Condition (6. 14).

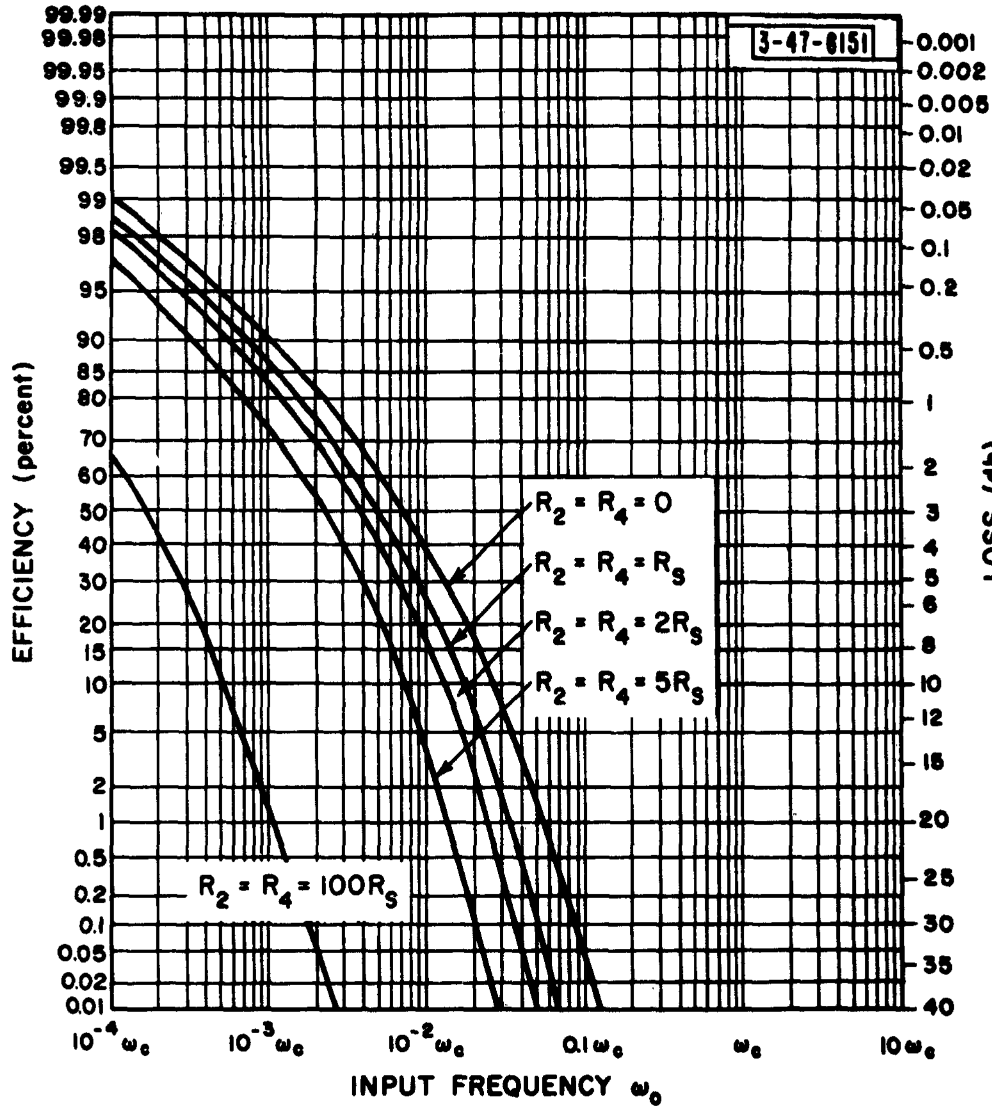

Fig. 6.1 Maximum efficiency of a 1-2-4-6 sextupler for several values of the idler resistances, R_2 and R_4.

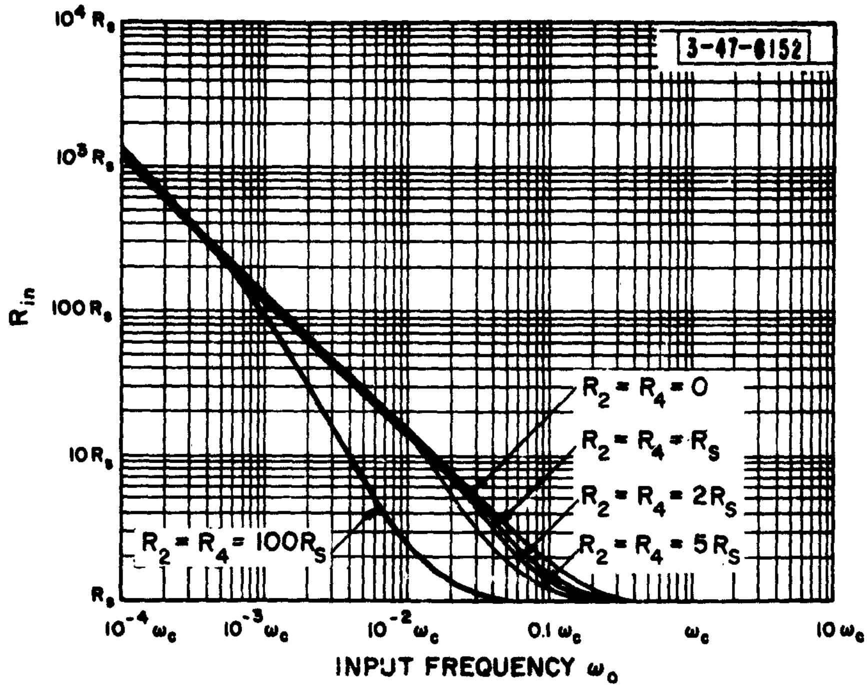

Fig. 6.2 Input resistance for maximum efficiency operation of a 1-2-4-6 sextupler for a variety of idler resistance, R_2 and R_4.

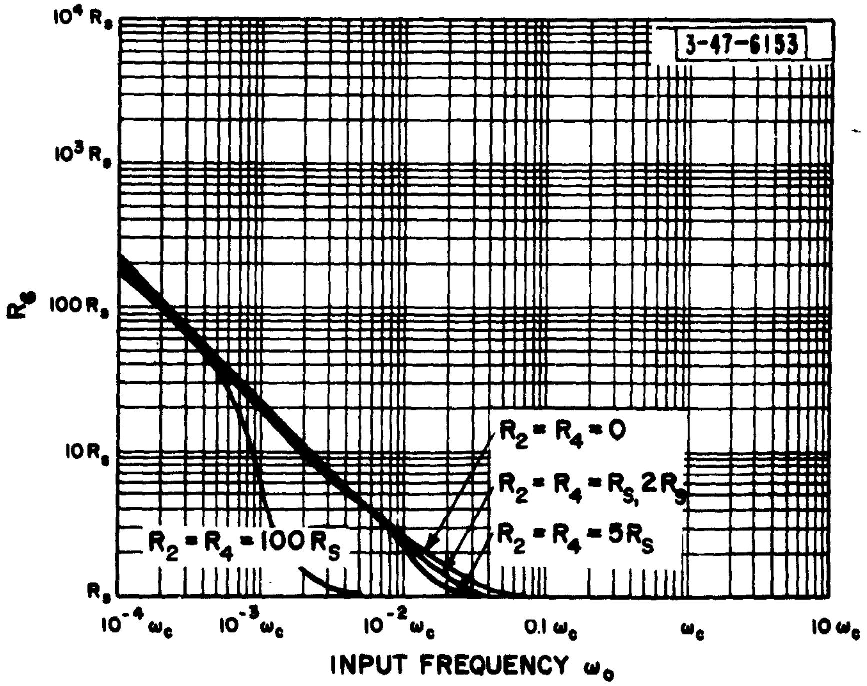

Fig. 6.3 Load resistance of a 1-2-4-6 sextupler adjusted for maximum efficiency operation.

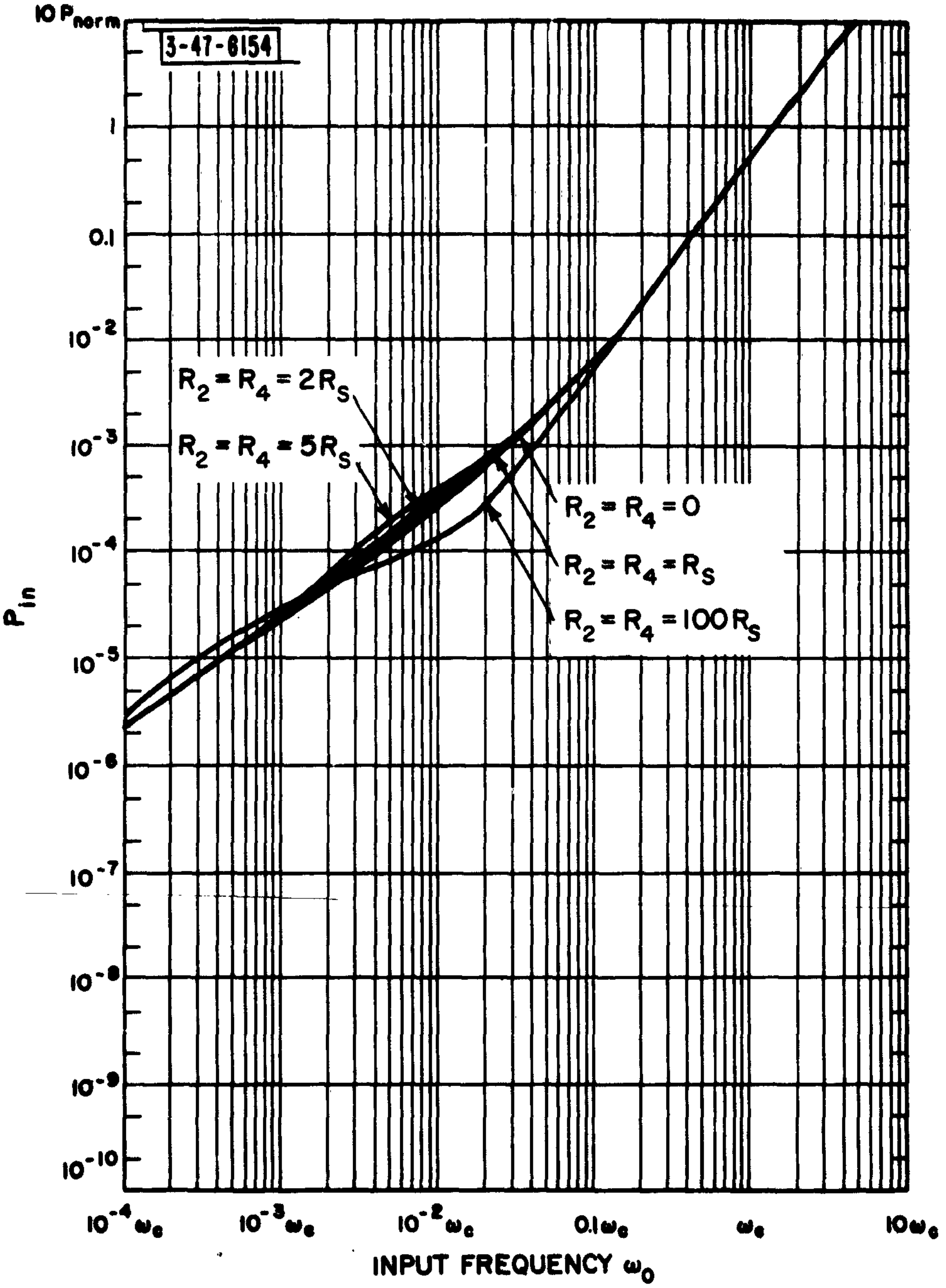

Fig. 6.4 Power input for maximum efficiency operation of a 1-2-4-6 sextupler for a variety of idler resistances, R_2 and R_4.

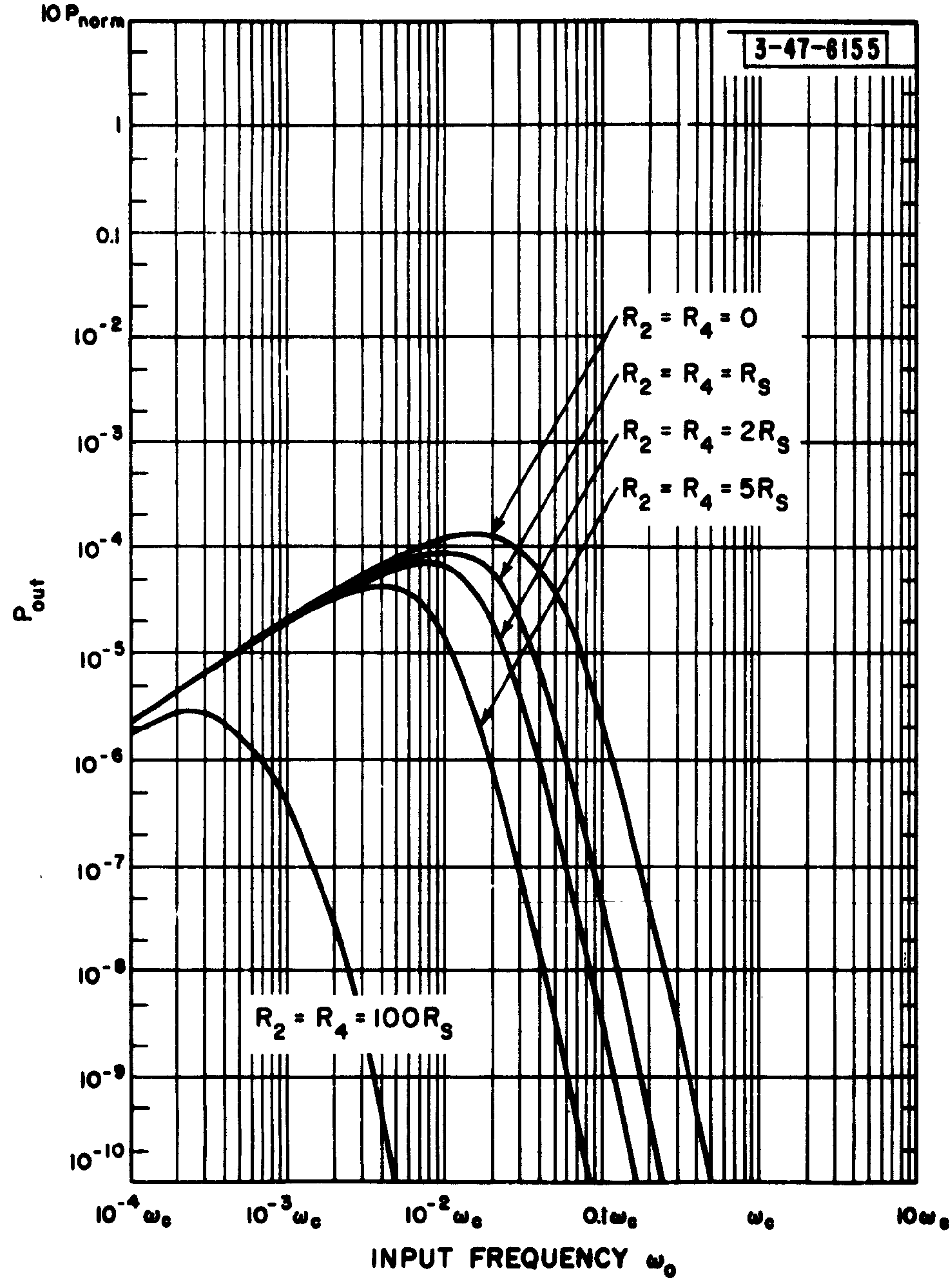

Fig. 6.5 Power output for maximum efficiency operation of a 1-2-4-6 sextupler for a variety of idler resistances, R_2 and R_4.

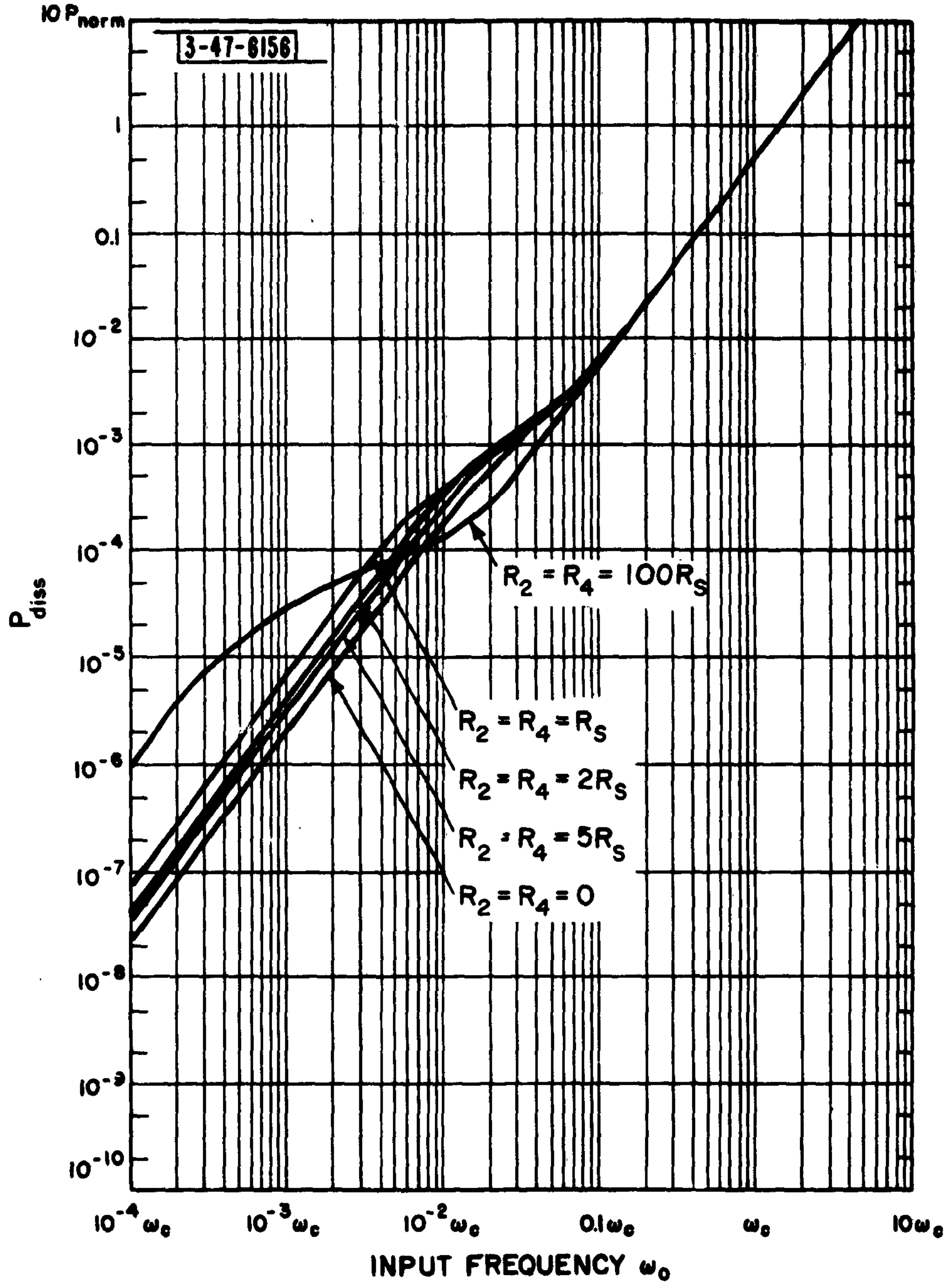

Fig. 6.6 Total power dissipated in a 1-2-4-6 sextupler adjusted for maximum efficiency operation.

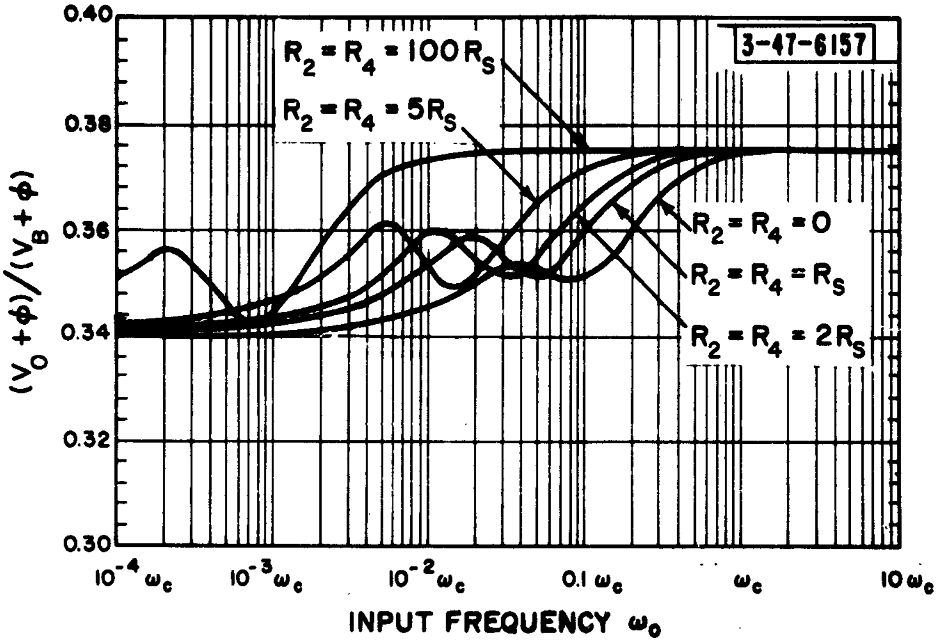

Fig. 6.7 Bias voltage for a 1-2-4-6 sextupler adjusted for maximum efficiency operation for a variety of idler resistances, R_2 and R_4.

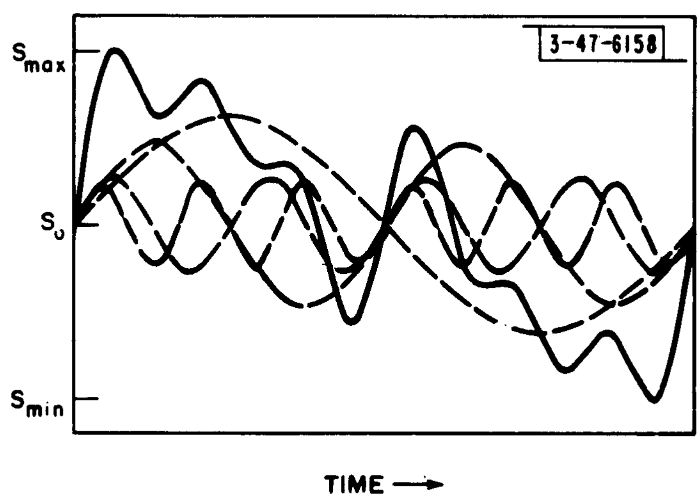

Fig. 6.8 Elastance waveform of a low-frequency 1-2-4-6 sextupler adjusted for maximum efficiency operation with lossless idler terminations.

6.3 Asymptotic Formulas for the 1-2-4-6 Sextupler

Like the preceding multipliers, the 1-2-4-6 sextupler behavior can be described by asymptotic formulas at low and high frequencies. The limiting values of the m_k for low frequencies, as found from the computed data, are used in Eqs. (6.4) to (6.13) to determine the appropriate formulas. For high frequencies we use the limiting values, $R_{in} \approx R_6 \approx R_s$, $m_1 \approx 0.25$, and $m_k \ll m_1$ for $k > 1$, in Eqs. (6.4) to (6.13) to find asymptotic relations. These formulas are summarized in Table 6.1 for the lossless idler case ($R_2 = R_4 = 0$). One set of asymptotic relations describes the high frequency performance, since efficiency and power output are simultaneously maximized. Both maximum efficiency and maximum power output relations are given for low frequencies.

6.4 1-2-3-6 Sextupler Formulas

The second idler in this sextupler is at the third harmonic, rather than the fourth harmonic, so the formulas are somewhat different from those given in Section 6.1 for the 1-2-4-6 sextupler. From Eq. (2.24) the idler and load resistance equations are

$$\frac{R_6 + R_s}{R_s} = \frac{\omega_c}{12\omega_o} \frac{(jM_3)^2}{jM_6} , \qquad (6.18)$$

$$\frac{R_3 + R_s}{R_s} = \frac{\omega_c}{3\omega_o} \frac{(jM_1)(jM_2) - (jM_3)^*(jM_6)}{jM_3} , \qquad (6.19)$$

$$\frac{R_2 + R_s}{R_s} = \frac{\omega_c}{4\omega_o} \frac{(jM_1)^2 - 2(jM_1)^*(jM_3)}{jM_2} . \qquad (6.20)$$

As before, we find it convenient to choose the time origin such that jM_1 is real and positive. Equation (6.18) shows that the phase angle of jM_6 is twice the phase angle of jM_3. Use of this information in Eq. (6.19) shows that the angle of jM_3 is equal to the angle of jM_2. Then, from Eq. (6.20), we find that jM_2 is real and positive. Therefore, each jM_k is real and positive and thus equal to its magnitude m_k.

Table 6.1 Asymptotic Formulas for the 1-2-4-6 Sextupler

Low-frequency and high-frequency formulas are given for the abrupt-junction-varactor sextupler with idlers at $2\omega_o$ and $4\omega_o$. We have assumed that S_{min} is negligible in comparison to S_{max}, and that $R_2 = R_4 = 0$.

	Low Frequency		High Frequency
	Maximum ϵ	Maximum P_{out}	Max. ϵ and P_{out}
ϵ	$1 - 99\,\frac{\omega_o}{\omega_c}$	$1 - 104\,\frac{\omega_o}{\omega_c}$	$9.10 \times 10^{-13}(\frac{\omega_c}{\omega_o})^{10}$
R_{in}	$0.117(\frac{\omega_c}{\omega_o})R_s$	$0.130(\frac{\omega_c}{\omega_o})R_s$	R_s
R_6	$0.135(\frac{\omega_c}{6\omega_o})R_s$	$0.110(\frac{\omega_c}{6\omega_o})R_s$	R_s
$\frac{P_{in}}{P'_{norm}}$	$0.0219(\frac{\omega_o}{\omega_c})$	$0.0225(\frac{\omega_o}{\omega_c})$	$0.500(\frac{\omega_o}{\omega_c})^2$
$\frac{P_{out}}{P'_{norm}}$	$0.0219(\frac{\omega_o}{\omega_c})$	$0.0225(\frac{\omega_o}{\omega_c})$	$4.55 \times 10^{-13}(\frac{\omega_c}{\omega_o})^8$
$\frac{P_{diss}}{P'_{norm}}$	$2.18(\frac{\omega_o}{\omega_c})^2$	$2.34(\frac{\omega_o}{\omega_c})^2$	$0.500(\frac{\omega_o}{\omega_c})^2$
$\frac{V_o + \varphi}{V_B + \varphi}$	0.340	0.342	0.375
m_1	0.153	0.147	0.250
m_2	0.117	0.131	$0.0156(\frac{\omega_c}{\omega_o})$
m_4	0.067	0.055	$3.05 \times 10^{-5}(\frac{\omega_c}{\omega_o})^3$
m_6	0.058	0.066	$3.97 \times 10^{-8}(\frac{\omega_c}{\omega_o})^5$

In terms of the m_k, the various 1-2-3-6 sextupler formulas are

$$R_6 = R_s\left(\frac{\omega_c}{12\omega_o}\,\frac{m_3^2}{m_6} - 1\right) , \tag{6.21}$$

$$R_3 = R_s\left(\frac{\omega_c}{3\omega_o}\,\frac{m_1 m_2 - m_3 m_6}{m_3} - 1\right) , \tag{6.22}$$

$$R_2 = R_s\left(\frac{\omega_c}{4\omega_o}\,\frac{m_1^2 - 2m_1 m_3}{m_2} - 1\right) , \tag{6.23}$$

$$R_{in} = R_s\left(\frac{\omega_c}{\omega_o}\,\frac{m_1 m_2 + m_2 m_3}{m_1} + 1\right) , \tag{6.24}$$

$$\frac{P_{in}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 \left[\frac{\omega_c}{\omega_o} m_1 m_2 (m_1 + m_3) + m_1^2\right] , \tag{6.25}$$

$$\frac{P_{out}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 \left(\frac{3\omega_c}{\omega_o} m_3^2 m_6 - 36 m_6^2\right) , \tag{6.26}$$

$$\frac{P_{diss}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 \left[\frac{\omega_c}{\omega_o}(m_1^2 m_2 + m_1 m_2 m_3 - 3m_3^2 m_6) + m_1^2 + 36 m_6^2\right] , \tag{6.27}$$

$$\frac{P_{diss,v}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 (m_1^2 + 4m_2^2 + 9m_3^2 + 36m_6^2) , \tag{6.28}$$

$$\epsilon = \frac{3m_6}{m_1}\,\frac{\frac{\omega_c}{\omega_o} m_3^2 - 12 m_6}{\frac{\omega_c}{\omega_o} m_2(m_1 + m_3) + m_1} , \tag{6.29}$$

$$\frac{V_o + \varphi}{V_B + \varphi} = \left(\frac{S_{max} - S_{min}}{S_{max}}\right)^2 \left[\frac{1}{4} + 2(m_1^2 + m_2^2 + m_3^2 + m_6^2)\right] + \frac{S_{min}}{S_{max}} \quad . \tag{6.30}$$

We have used P'_{norm}, as defined by Eq. (3.12), to normalize the power relations, and we have set $m_o = 1/2$ to write the bias voltage in the form given by Eq. (6.30).

The M_k are imaginary, so Condition (2.17) can be used as the bound on the magnitudes of the m_k:

$$m_1 \sin \omega_o t + m_2 \sin 2\omega_o t + m_3 \sin 3\omega_o t + m_6 \sin 6\omega_o t \leq 0.25 \tag{6.31}$$

for all values of t.

6.5 Solution of the 1-2-3-6 Sextupler Equations

The technique for solving these equations is similar to that used for the 1-2-4-6 sextupler. If m_1, m_2, m_3, and m_6 are known, then all quantities of interest can be calculated. More often, however, the m_k must be computed for specified values of R_2, R_3, R_6, and ω_o/ω_c such that Condition (6.31) is satisfied. Usually we must find the loading conditions required for maximum efficiency or maximum power output operation. These maxima occur when the varactor is fully driven, that is, when the elastance attains the values S_{min} and S_{max} one or more times during each cycle. Thus, the problem is to find values for R_2, R_3, and R_6 such that efficiency or power output is maximized. The values of the m_k must be compatible with the load and idler resistances, and they must be such that Condition (6.31) is satisfied with the equality sign at the time, t_o, when m(t) is a maximum.

The 1-2-3-6 sextupler equations, like those for the other abrupt-junction-varactor multipliers, are solved by an iterative numerical procedure. Usually, we choose values for the idler resistances, the load resistance, and the frequency and then compute the required values for the m_k. It is, therefore, convenient to solve Eqs. (6.21), (6.22), and (6.23) for three of the m_k in terms of the other one. If we choose m_1 as the reference, then

$$\left(\frac{m_3}{m_1}\right)^3 + \frac{m_3}{m_1}\left(\frac{36\omega_o^2}{m_1^2 \omega_c^2} \frac{R_3 + R_s}{R_s} \frac{R_6 + R_s}{R_s} + 6\frac{R_6 + R_s}{R_2 + R_s}\right) - 3\frac{R_6 + R_s}{R_2 + R_s} = 0 , \tag{6.32}$$

$$\frac{m_2}{m_1} = \frac{m_1 \omega_c}{4\omega_o} \frac{R_s}{R_2 + R_s} \left(1 - 2\frac{m_3}{m_1}\right) , \tag{6.33}$$

$$\frac{m_6}{m_1} = \frac{m_1 \omega_c}{12\omega_o} \frac{R_s}{R_6 + R_s} \left(\frac{m_3}{m_1}\right)^2 . \tag{6.34}$$

The cubic equation for m_3 has only one real root which lies between zero and $\frac{1}{2} m_1$. Some other m_k could be chosen as the reference, but it is apparently not possible to avoid a cubic equation. The choice of m_1 as a reference seems to be preferable, since m_1 goes to the fixed limit of 0.25 at high frequencies which is convenient when solutions are obtained by iterative methods.

The above equations have been solved numerically on an I. B. M. 7090 digital computer for maximum efficiency and maximum power output operation. Efficiency, power input, and power output as functions of R_6 for a given frequency and specific values of R_2 and R_3 are found to look somewhat like Fig. 3.1 for the tripler. Thus, for practical purposes, efficiency and power output may be simultaneously maximized (at least for small values of the idler resistances).

The computed results for maximum efficiency operation are presented in Figs. 6.9 to 6.15. In these figures we show efficiency, input resistance, load resistance, power input, power output, dissipated power, and bias voltage as functions of frequency for several values of R_2 and R_3. These plots are similar to those given for the other multipliers and are interpreted in the same way.

The minimum elastance S_{min} has been neglected in Figs. 6.12 to 6.15. When S_{min} is not small in comparison with S_{max}, P'_{norm} must be used as the normalization power. Also, the values given in Fig. 6.15 for the bias

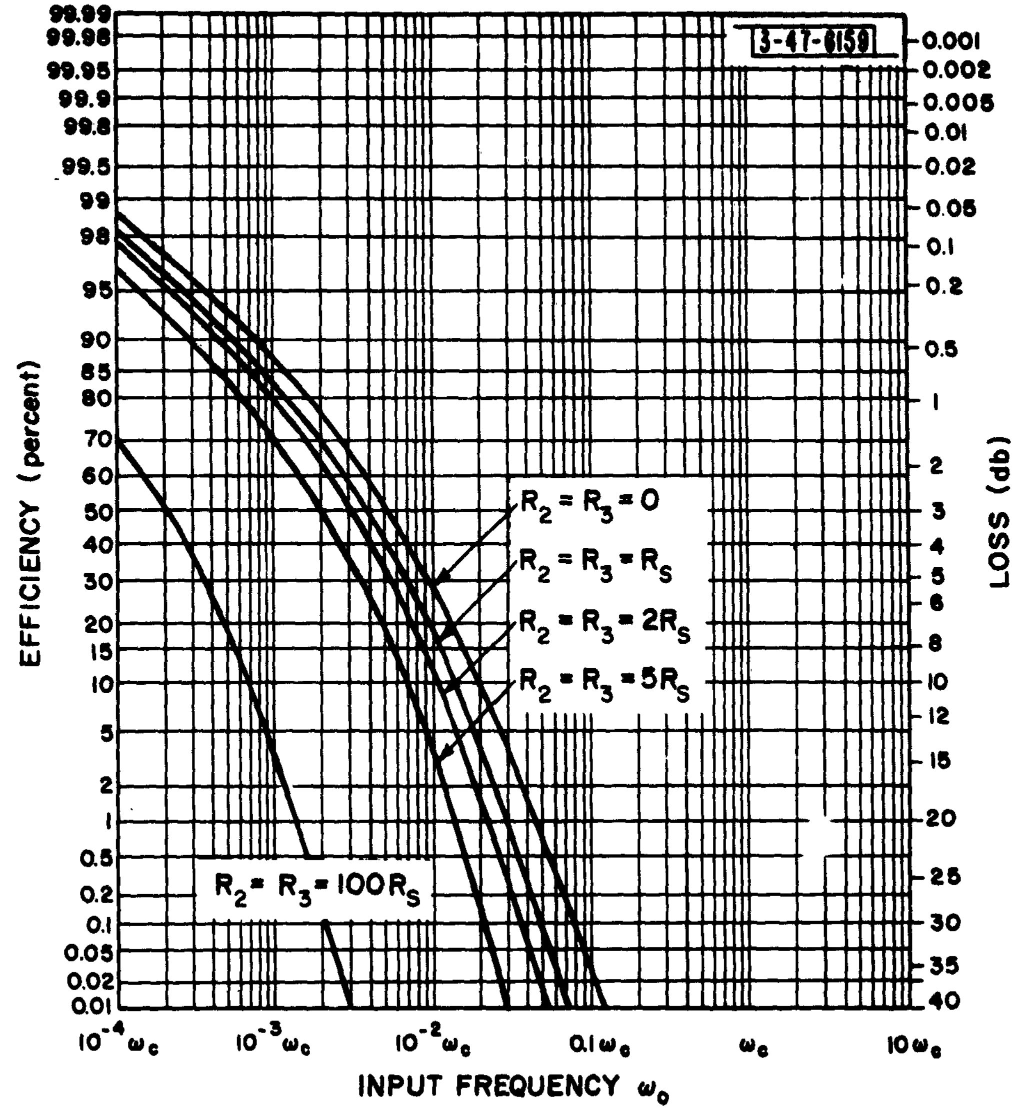

Fig. 6.9 Maximum efficiency of a 1-2-3-6 sextupler for several values of the idler resistances, R_2 and R_3.

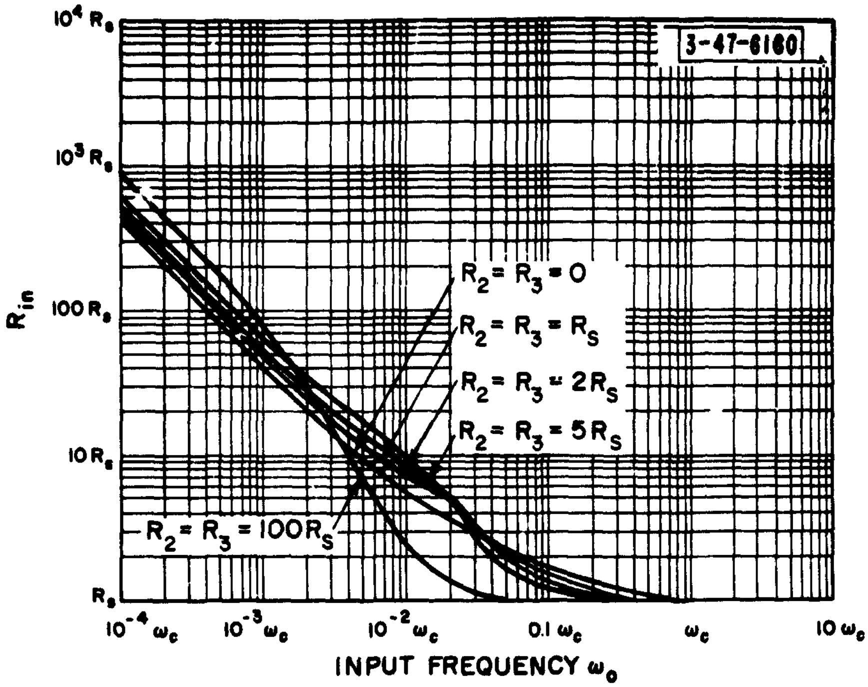

Fig. 6.10 Input resistance of a 1-2-3-6 sextupler adjusted for maximum efficiency operation.

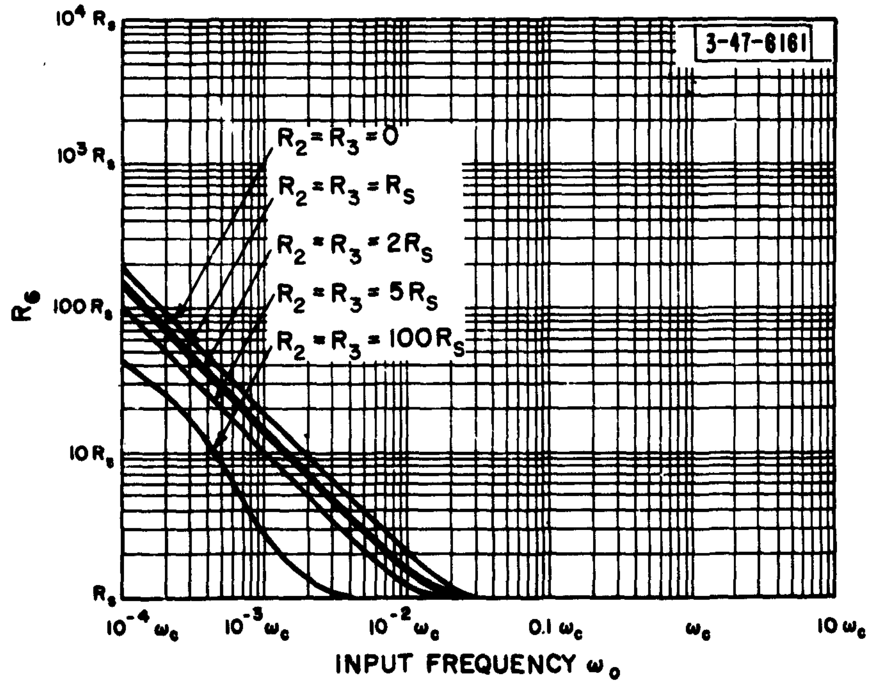

Fig. 6.11 Load resistance for maximum efficiency operation of a 1-2-3-6 sextupler for a variety of idler resistances, R_2 and R_3.

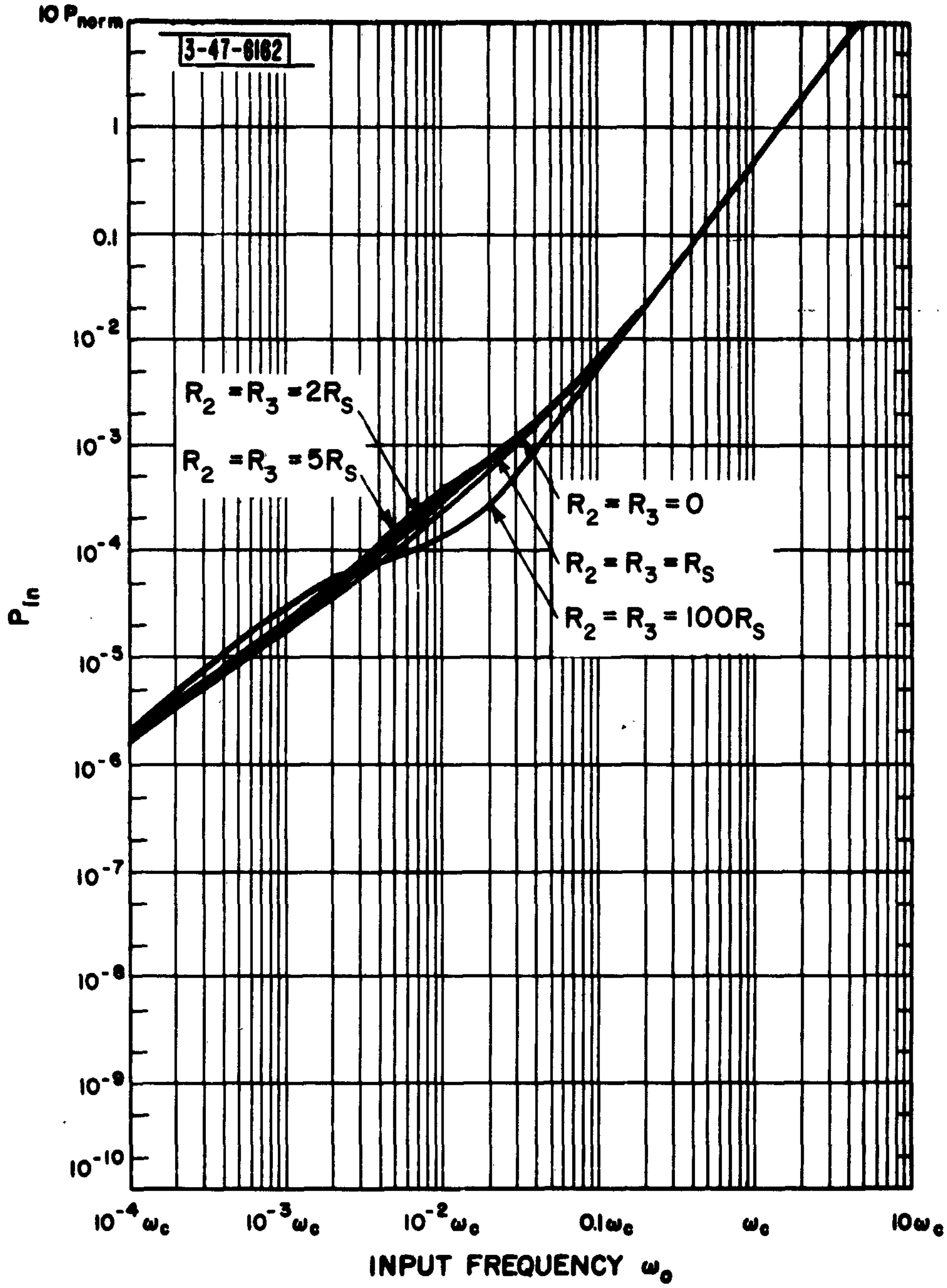

Fig. 6.12 Power input for maximum efficiency operation of a 1-2-3-6 sextupler for a variety of idler resistances, R_2 and R_3.

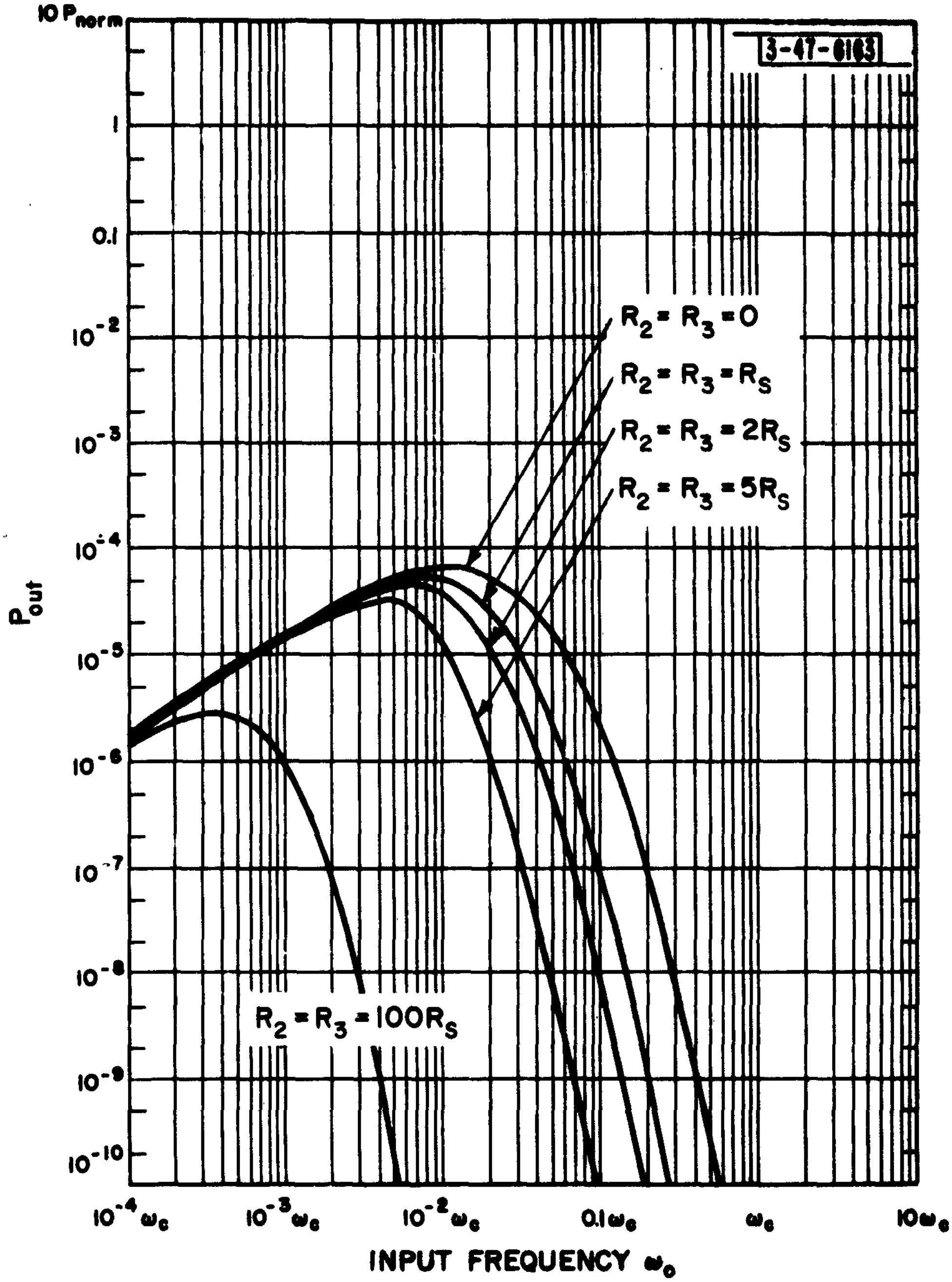

Fig. 6.13 Power output of a 1-2-3-6 sextupler adjusted for maximum efficiency operation.

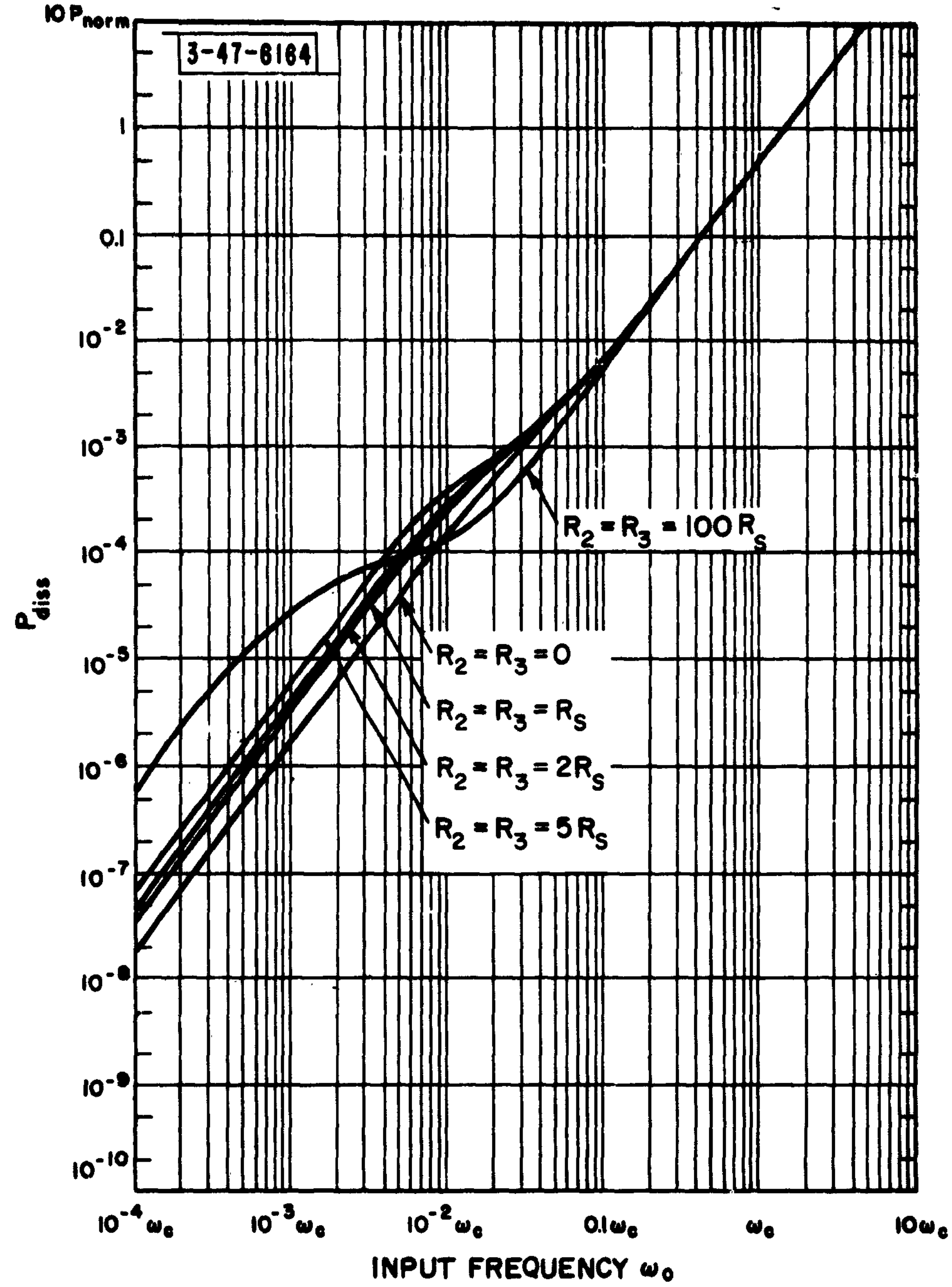

Fig. 6.14 Total power dissipated in a 1-2-3-6 sextupler adjusted for maximum efficiency operation.

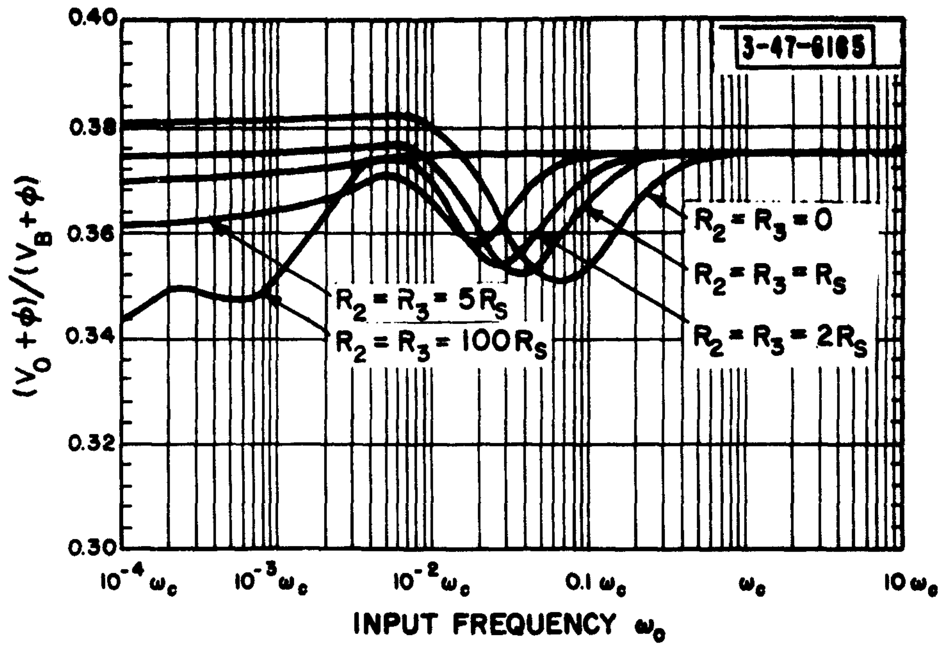

Fig. 6.15 Bias voltage for a 1-2-3-6 sextupler adjusted for maximum efficiency operation for a variety of idler resistances, R_2 and R_3.

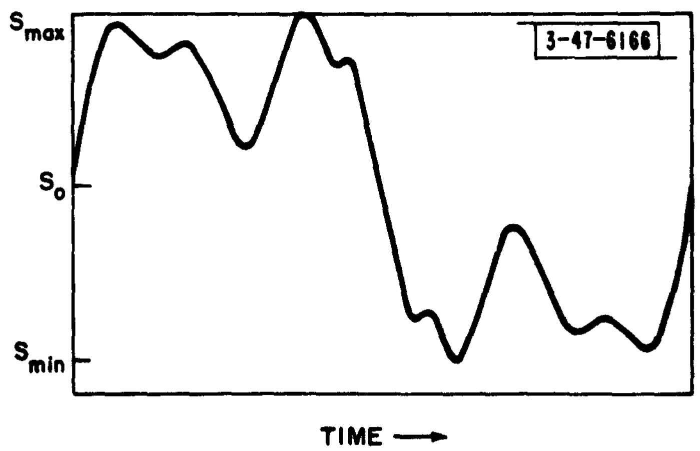

Fig. 6.16 Elastance waveform of a low-frequency 1-2-3-6 sextupler adjusted for maximum efficiency operation with lossless idler terminations.

voltage must be multiplied by the factor $(S_{max} - S_{min})^2/S_{max}^2$ and then added to S_{min}/S_{max} to find the bias voltage when S_{min} is not negligible. Since the solutions are for maximum drive, the average elastance is given by

$$S_o = \frac{S_{max} + S_{min}}{2} \quad . \tag{6.35}$$

The conditions for maximum power output are very similar to those for maximum efficiency, at least for small values of R_2 and R_3. For large idler resistances the optimizations are quite different, but we do not plot the results because they are of little practical interest.

A plot of the elastance waveform for maximum efficiency operation with lossless idlers at low frequencies is shown in Fig. 6.16. For maximum drive S_{min} and S_{max} are attained at least once per cycle. With some loading conditions, it is possible to attain S_{min} and S_{max} twice per cycle. Therefore, care must be taken in performing the calculations to make sure that the highest maximum is used in checking Condtion (6.31).

6.6 Asymptotic Formulas for the 1-2-3-6 Sextupler

The performance of the 1-2-3-6 sextupler at low and high frequencies can be described by asymptotic formulas. The low-frequency formulas are found by using the limiting values of the m_k in Eqs. (6.21) to (6.30). At high frequencies the input and load resistances both become nearly equal to the series resistance of the varactor, while the m_k become small, except m_1 which approaches 0.25. These formulas are summarized in Table 6.2 for the lossless idler case ($R_2 = R_3 = 0$). One set of asymptotic relations describes the high frequency performance, since efficiency and power output are simultaneously maximized. Both maximum efficiency and maximum power output relations are given for low frequencies.

6.7 Comparison of the 1-2-4-6 and 1-2-3-6 Sextuplers

Since there are two practical idler configurations for the abrupt-junction-varactor sextupler, it is of interest to compare their performance. Most of the comparisons can be made by referring to Figs. 6.1 to 6.7 and Table 6.1 and to Figs. 6.9 to 6.15 and Table 6.2.

Table 6.2 Asymptotic Formulas for the 1-2-3-6 Sextupler

Low-frequency and high-frequency formulas are given for the abrupt-junction-varactor sextupler with idlers at $2\omega_o$ and $3\omega_o$. We have assumed that S_{min} is negligible in comparison to S_{max}, and that $R_2 = R_3 = 0$.

	Low Frequency		High Frequency
	Maximum ϵ	Maximum P_{out}	Max. ϵ and P_{out}
ϵ	$1 - 134(\frac{\omega_o}{\omega_c})$	$1 - 154\frac{\omega_o}{\omega_c}$	$2.87 \times 10^{-12}(\frac{\omega_c}{\omega_o})^{10}$
R_{in}	$0.041(\frac{\omega_c}{\omega_o})R_s$	$0.059(\frac{\omega_c}{\omega_o})R_s$	R_s
R_6	$0.114(\frac{\omega_c}{6\omega_o})R_s$	$0.062(\frac{\omega_c}{6\omega_o})R_s$	R_s
$\frac{P_{in}}{P'_{norm}}$	$0.0162(\frac{\omega_o}{\omega_c})$	$0.0183(\frac{\omega_o}{\omega_c})$	$0.500(\frac{\omega_o}{\omega_c})^2$
$\frac{P_{out}}{P'_{norm}}$	$0.0162(\frac{\omega_o}{\omega_c})$	$0.0183(\frac{\omega_o}{\omega_c})$	$1.44 \times 10^{-12}(\frac{\omega_c}{\omega_o})^8$
$\frac{P_{diss}}{P'_{norm}}$	$2.17(\frac{\omega_o}{\omega_c})^2$	$2.83(\frac{\omega_o}{\omega_c})^2$	$0.500(\frac{\omega_o}{\omega_c})^2$
$\frac{V_o + \varphi}{V_B + \varphi}$	0.381	0.363	0.375
m_1	0.222	0.198	0.250
m_2	0.027	0.039	$0.0156(\frac{\omega_c}{\omega_o})$
m_3	0.111	0.099	$0.0013(\frac{\omega_c}{\omega_o})^2$
m_6	0.054	0.078	$7.06 \times 10^{-8}(\frac{\omega_c}{\omega_o})^5$

The maximum efficiencies with lossless idlers are shown in Fig. 6.17 for comparison. Higher efficiency is obtainable from the 1-2-4-6 sextupler for frequencies up to approximately $0.1\omega_c$, although the difference is not very large. The maximum difference in efficiency is about 11 per cent. For very high frequencies (ω_o greater than about $0.1\omega_c$) the 1-2-3-6 sextupler gives higher efficiency, although this fact is not evident in Fig. 6.17 because both efficiencies are so low. (This may be appreciated from a comparison of the high-frequency relations given in Tables 6.1 and 6.2.)

Also of considerable interest is the comparative power handling capability of the two sextuplers. The power outputs at maximum efficiency with lossless idlers are shown in Fig. 6.18 for comparison. The 1-2-4-6 sextupler has the highest power output for frequencies up to approximately $0.1\omega_c$, while at very high frequencies the 1-2-3-6 sextupler delivers the most power. At low frequencies the ratio of the power outputs, $P_{out}(1\text{-}2\text{-}3\text{-}6)/P_{out}(1\text{-}2\text{-}4\text{-}6)$, is C.74, while at high frequencies this ratio becomes 3.16.

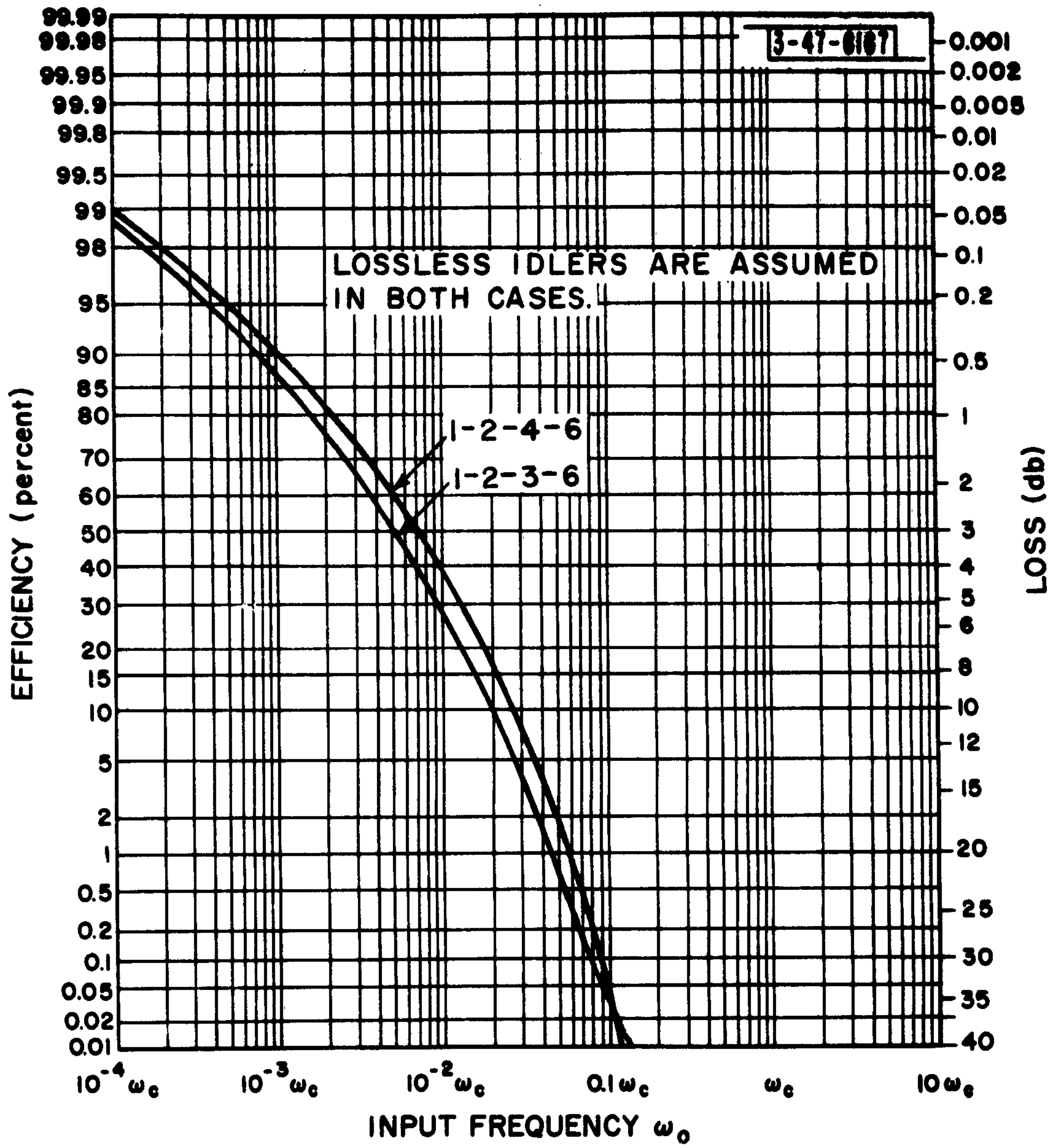

Fig. 6.17 Comparison of the maximum efficiency obtainable from the abrupt-junction-varactor 1-2-4-6 and 1-2-3-6 sextuplers. In both cases the idler terminations are lossless.

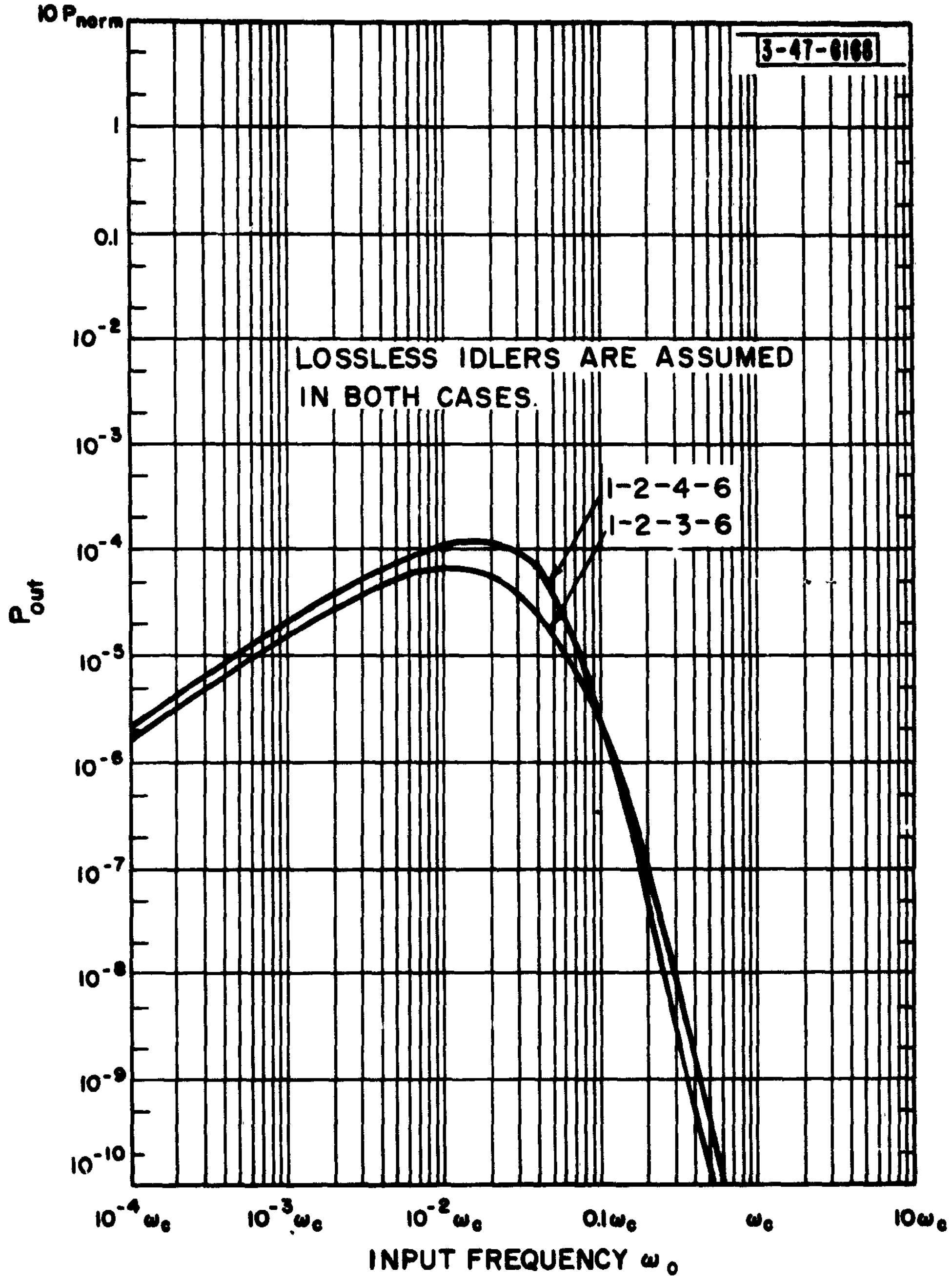

Fig. 6.18 Comparison of the power output for maximum efficiency operation with lossless idler terminations for the abrupt-junction-varactor 1-2-4-6 and 1-2-3-6 sextuplers.

VII. OCTUPLER SOLUTION

There is only one possible two-idler configuration for an octupler. It is one with idler currents flowing at the second and fourth harmonics, and is called a 1-2-4-8 octupler. Obviously, there are many multiple-idler configurations which would allow multiplication-by-eight, but we will not go into this problem because it becomes extremely complicated. The 1-2-4-8 octupler is relatively easy to analyze and is, therefore, the one which we will discuss, although it is not necessarily the best possible octupler.

7.1 1-2-4-8 Octupler Formulas

The formulas in Chapter II apply to the octupler with all M_k equal to zero except for $k = 0, \pm 1, \pm 2, \pm 4,$ and ± 8. The i ler and load resistance equations are

$$\frac{R_8 + R_s}{R_s} = \frac{\omega_c}{16\omega_o} \frac{(jM_4)^2}{jM_8} \quad , \tag{7.1}$$

$$\frac{R_4 + R_s}{R_s} = \frac{\omega_c}{8\omega_o} \frac{(jM_2)^2 - 2(jM_4)^*(jM_8)}{jM_4} \quad , \tag{7.2}$$

$$\frac{R_2 + R_s}{R_s} = \frac{\omega_c}{4\omega_o} \frac{(jM_1)^2 - 2(jM_2)^*(jM_4)}{jM_2} \quad . \tag{7.3}$$

Equation (7.1) shows that the phase angle of jM_8 is twice the angle of jM_4. Then, Eq. (7.2) indicates that the angle of jM_4 is twice that of jM_2. When this information is used in Eq. (7 3), we find that the phase angle of jM_2 is twice the phase angle of jM_1. We can without a loss of generality assume that jM_1 is real and positive. Then each jM_k is real and positive and thus equal to its magnitude m_k, i.e., $jM_1 = m_1$, $jM_2 = m_2$, $jM_4 = m_4$, and $jM_8 = m_8$.

In terms of the m_k the various 1-2-4-8 octupler formulas are

$$R_8 = R_s\left(\frac{\omega_c}{16\omega_o}\ \frac{m_4^2}{m_8} - 1\right) , \tag{7.4}$$

$$R_4 = R_s\left(\frac{\omega_c}{8\omega_o}\ \frac{m_2^2 - 2m_4m_8}{m_4} - 1\right) , \tag{7.5}$$

$$R_2 = R_s\left(\frac{\omega_c}{4\omega_o}\ \frac{m_1^2 - 2m_2m_4}{m_2} - 1\right) , \tag{7.6}$$

$$R_{in} = R_s\left(\frac{\omega_c}{\omega_o} m_2 + 1\right) , \tag{7.7}$$

$$\frac{P_{in}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 \left(\frac{\omega_c}{\omega_o} m_1^2 m_2 + m_1^2\right) , \tag{7.8}$$

$$\frac{P_{out}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 \left(4 \frac{\omega_c}{\omega_o} m_4^2 m_8 - 64m_8^2\right) , \tag{7.9}$$

$$\frac{P_{diss}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 \left[\frac{\omega_c}{\omega_o} (m_1^2 m_2 - 4m_4^2 m_8) + m_1^2 + 64m_8^2\right] , \tag{7.10}$$

$$\frac{P_{diss,v}}{P'_{norm}} = 8\left(\frac{\omega_o}{\omega_c}\right)^2 (m_1^2 + 4m_2^2 + 16m_4^2 + 64m_8^2) , \tag{7.11}$$

$$\epsilon = \frac{4m_8}{m_1^2}\ \frac{\frac{\omega_c}{\omega_o} m_4^2 - 16m_8}{\frac{\omega_c}{\omega_o} m_2 + 1} , \tag{7.12}$$

$$\frac{V_o + \varphi}{V_B + \varphi} = \left(\frac{S_{max} - S_{min}}{S_{max}}\right)^2 \left[\frac{1}{4} + 2(m_1^2 + m_2^2 + m_4^2 + m_8^2)\right] + \frac{S_{min}}{S_{max}} . \tag{7.13}$$

We have set m_o equal to one-half in order to write the bias voltage in the form given by Eq. (7.13). The power relations have been normalized with respect to P'_{norm}, which is defined by Eq. (3.12).

We can use Condition (2.17) as the bound on the magnitudes of the m_k, since all of the M_k have been shown to be imaginary. The required condition for this specific case is

$$m_1 \sin \omega_o t + m_2 \sin 2\omega_o t + m_4 \sin 4\omega_o t + m_8 \sin 8\omega_o t \leq 0.25 \quad (7.14)$$

for all values of t.

7.2 Solution of the 1-2-4-8 Octupler Equations

The technique of solution of these equations is similar to that for the other multipliers. If values for the m_k are known, then all quantities of interest can be calculated. On the other hand the m_k values must be calculated when other information is available. The usual problem is to find the loading conditions required for maximum efficiency or maximum power output operation with the varactor fully driven. For the solution of this problem we usually begin by choosing values for R_2, R_4, and R_8 and then compute the required values for the m_k such that Condition (7.14) is satisfied with an equality at the time when m(t) is a maximum. The load resistance is then varied and the calculations repeated until the desired maximum has been found.

Since we are usually asked to find values for the m_k which are compatible with specific values of R_2, R_4, and R_8, it is convenient to solve Eqs. (7.4), (7.5), and (7.6) for three of the m_k in terms of the other. If m_4 is chosen as the reference, the simplest equations result:

$$\frac{m_8}{m_4} = \frac{m_4 \omega_c}{16\omega_o} \frac{R_s}{R_8 + R_s} , \quad (7.15)$$

$$\left(\frac{m_2}{m_4}\right)^2 = \frac{8\omega_o}{m_4 \omega_c} \frac{R_4 + R_s}{R_s} + \frac{m_4 \omega_c}{8\omega_o} \frac{R_s}{R_8 + R_s} , \quad (7.16)$$

$$\left(\frac{m_1}{m_4}\right)^2 = \frac{m_2}{m_4}\left(\frac{4\omega_o}{m_4\omega_c}\ \frac{R_2 + R_s}{R_s} + 2\right) . \tag{7.17}$$

These formulas, together with a specification of the drive level, are sufficient to determine the m_k uniquely. In the usual case the drive level is specified to be such that the elastance attains the values S_{min} and S_{max} one or more times during each cycle.

The above equations have been solved numerically on an I. B. M. 7090 digital computer for maximum efficiency and maximum power output operation. As was the case for the other multipliers, the 1-2-4-8 octupler can be, for practical purposes, maximized simultaneously for efficiency and power output (at least for small values of the idler resistances).

The computed results for maximum efficiency operation are presented in Figs. 7. 1 to 7. 7. In these figures we show efficiency, input resistance, load resistance, power input, power output, dissipated power, and bias voltage as functions of frequency for several values of the idler resistances, R_2 and R_4. These plots are similar to those given for the other multipliers and are interpreted in the same way.

The minimum elastance, S_{min}, has been neglected in Figs. 7. 4 to 7. 7. When S_{min} is not negligible, we must replace the normalization power P_{norm} with P'_{norm}. The bias voltage values as given in Fig. 7. 7 must be modified according to Eq. (7. 13) when S_{min} is important. Since the solutions are for maximum drive, the average elastance is given by Eq. (6. 35).

The conditions for maximum power output are very similar to those for maximum efficiency, at least for small values of R_2 and R_4. For large idler resistances the optimizations are quite different, but we do not plot the results because they are of little practical interest.

A plot of the elastance waveform for maximum efficiency operation with lossless idlers at low frequencies is shown in Fig. 7. 8. Like several of the other higher-order multipliers, the elastance waveform for the octupler may attain the values of S_{min} and S_{max} more than once per cycle. Therefore, the calculations must be carefully performed to make sure that the highest maximum is used in checking Condition (7. 14).

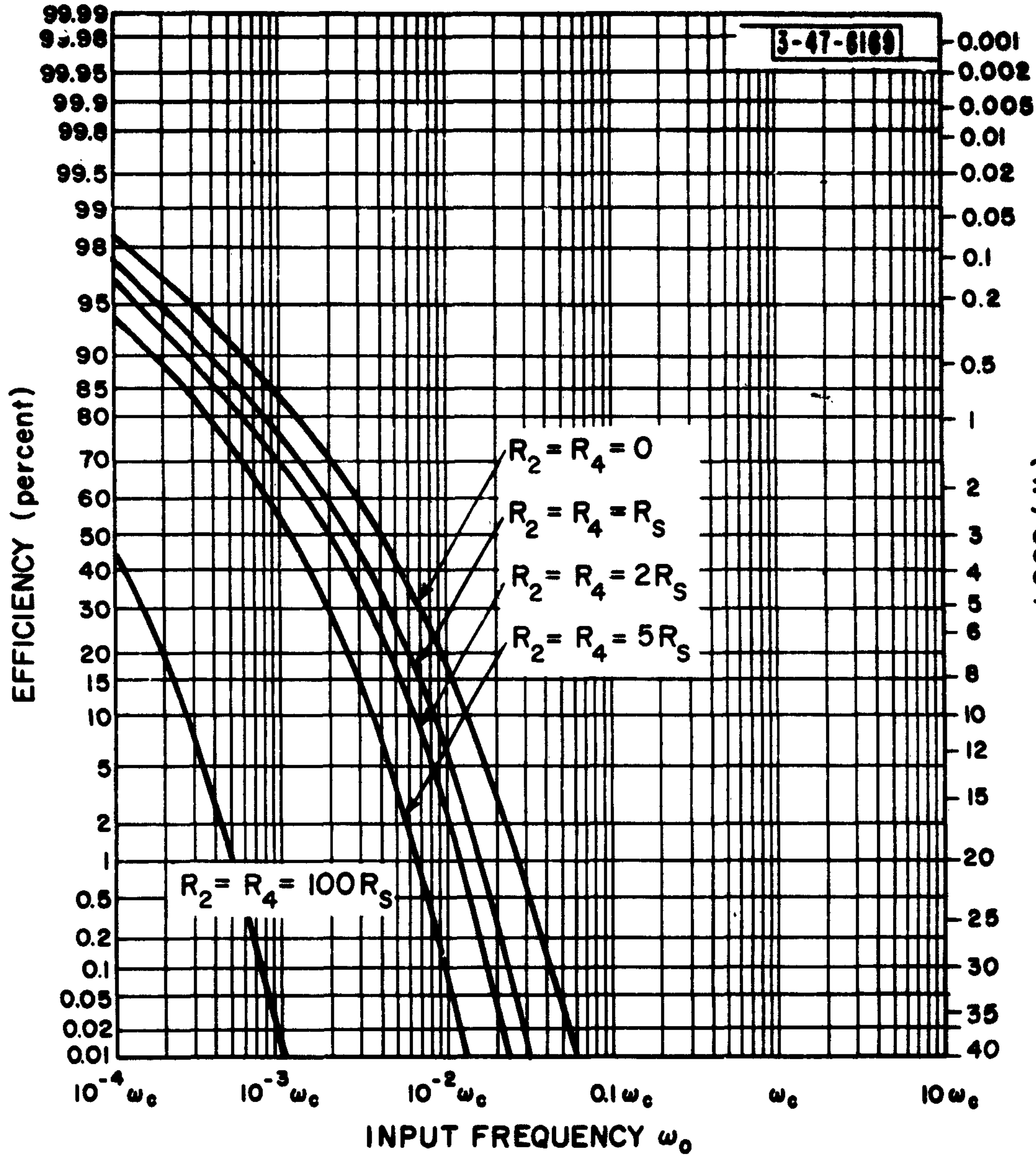

Fig. 7.1 Maximum efficiency of an abrupt-junction-varactor octupler for several values of the idler resistances, R_2 and R_4.

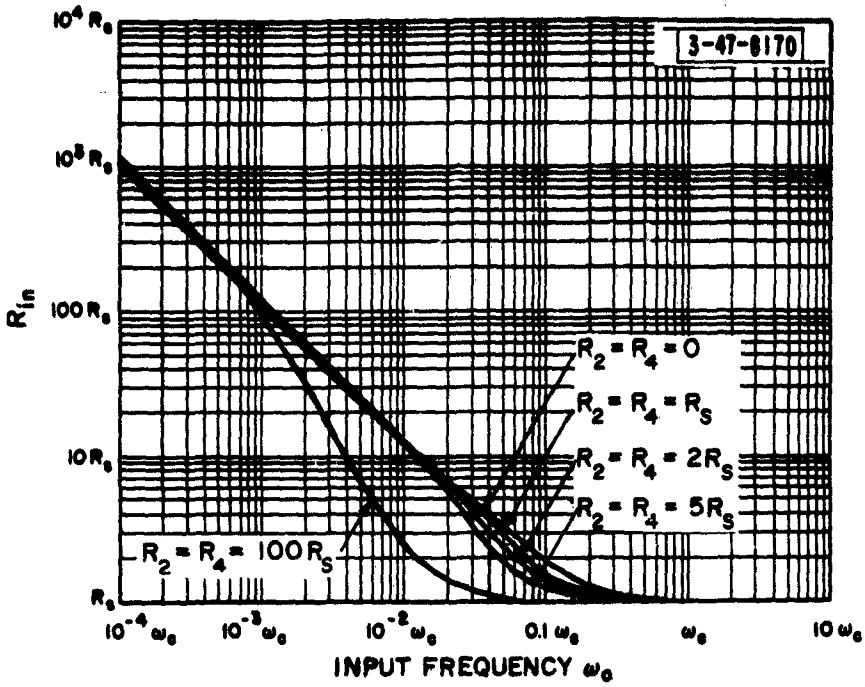

Fig. 7.2 Input resistance of a 1-2-4-8 octupler adjusted for maximum efficiency operation for a variety of idler resistances, R_2 and R_4.

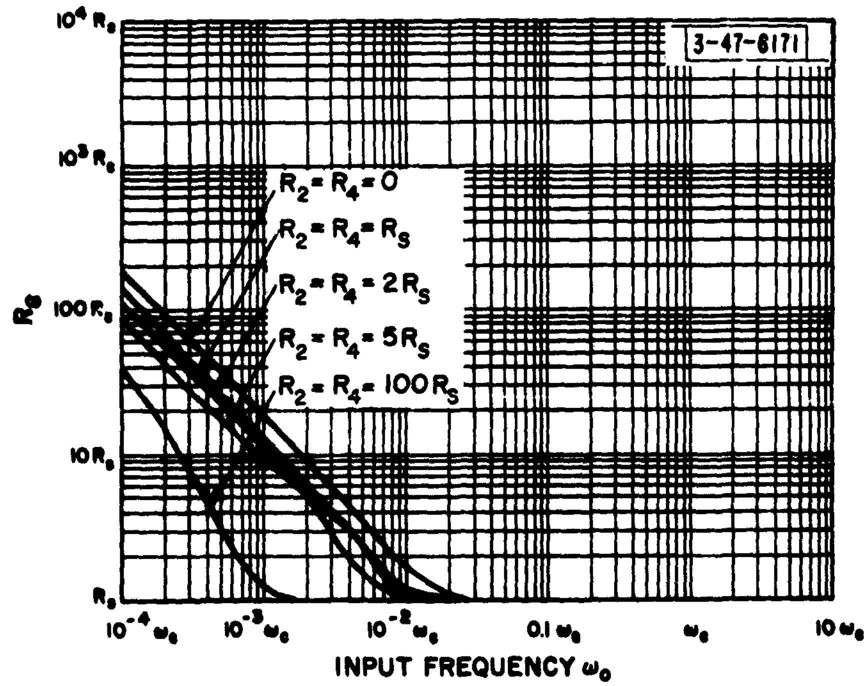

Fig. 7.3 Load resistance for maximum efficiency operation of a 1-2-4-8 octupler.

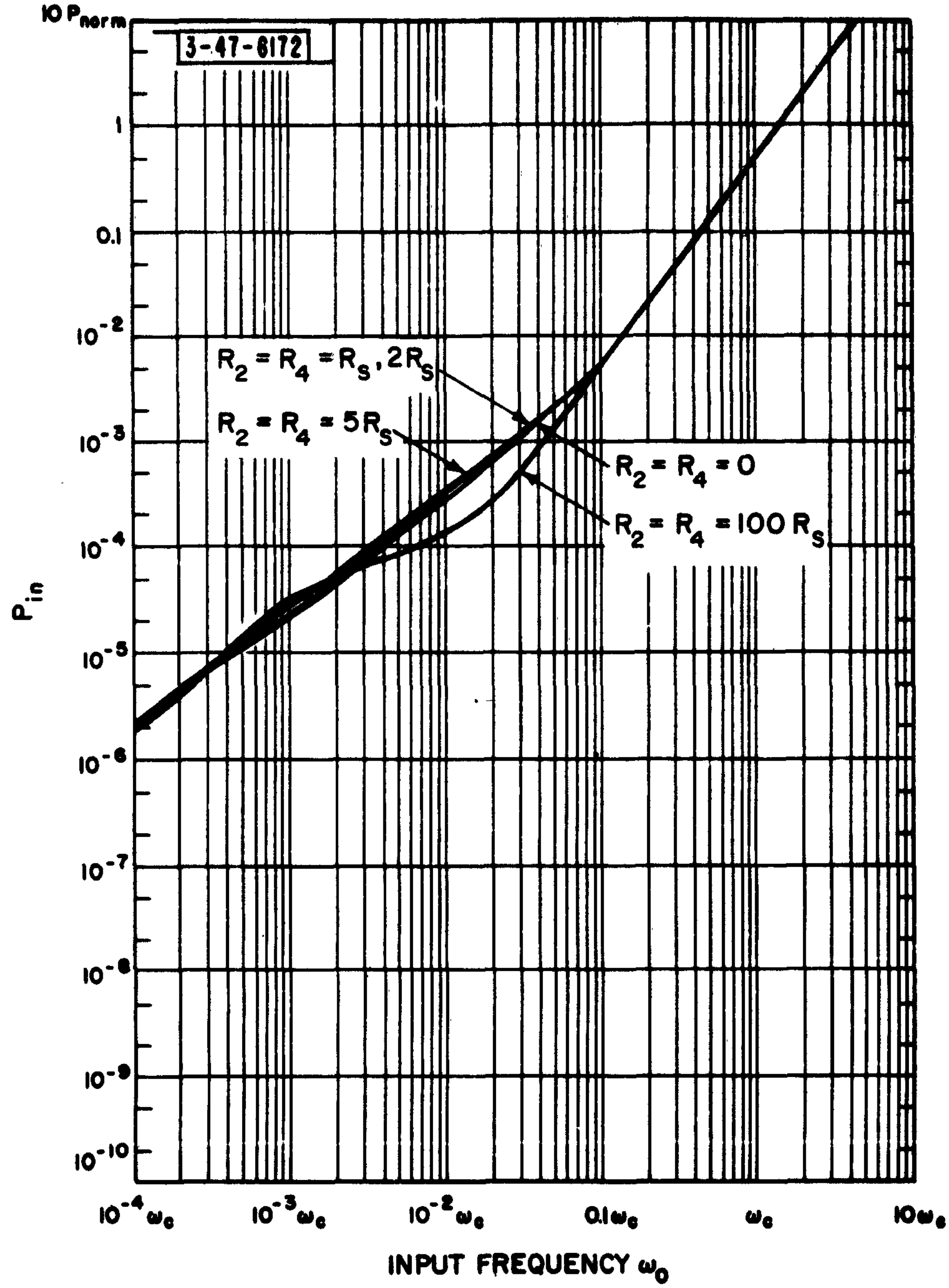

Fig. 7.4 Power input for maximum efficiency operation of a 1-2-4-8 octupler for a variety of idler resistances, R_2 and R_4.

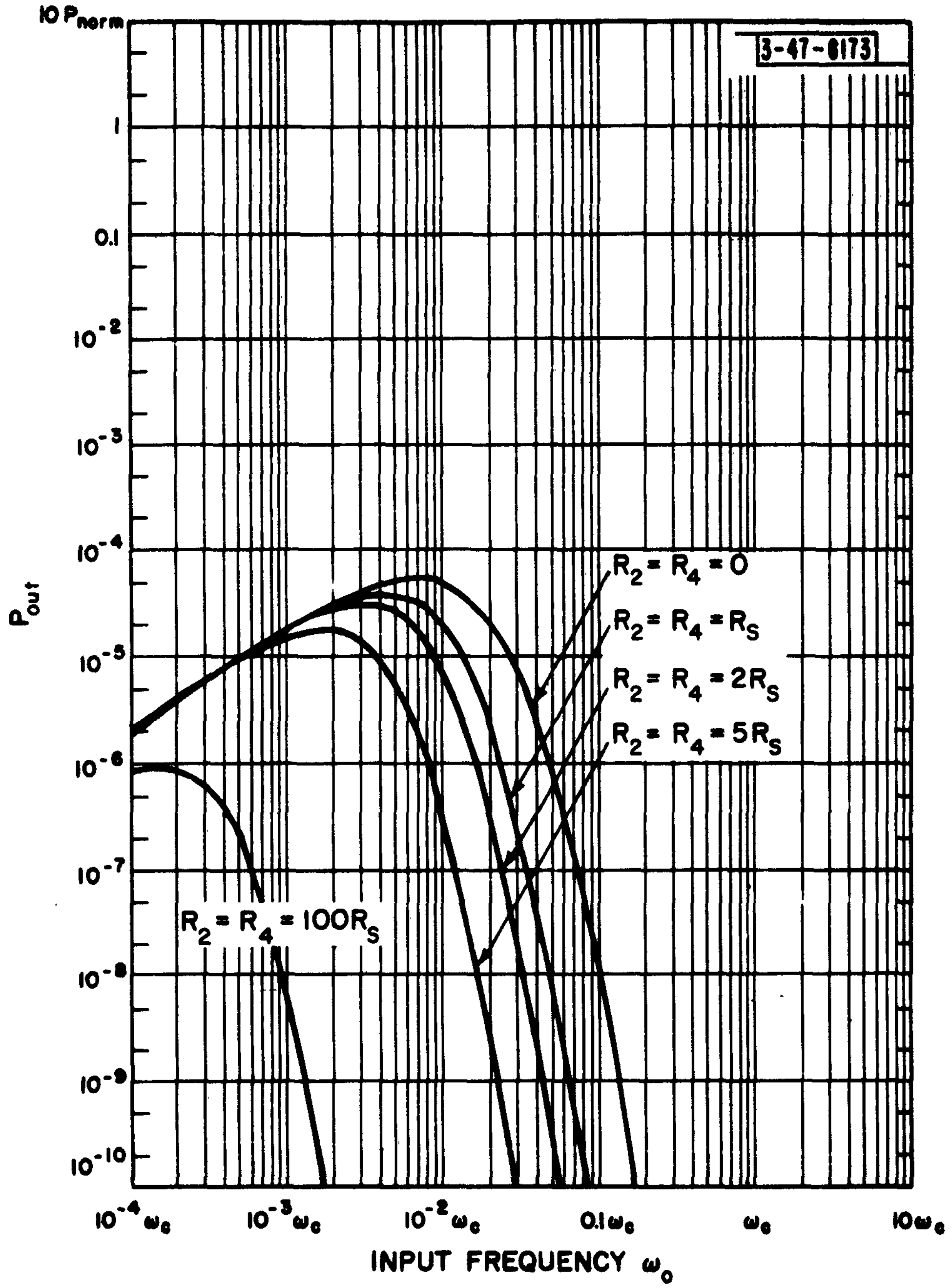

Fig. 7.5 Power output of a 1-2-4-8 octupler adjusted for maximum efficiency operation.

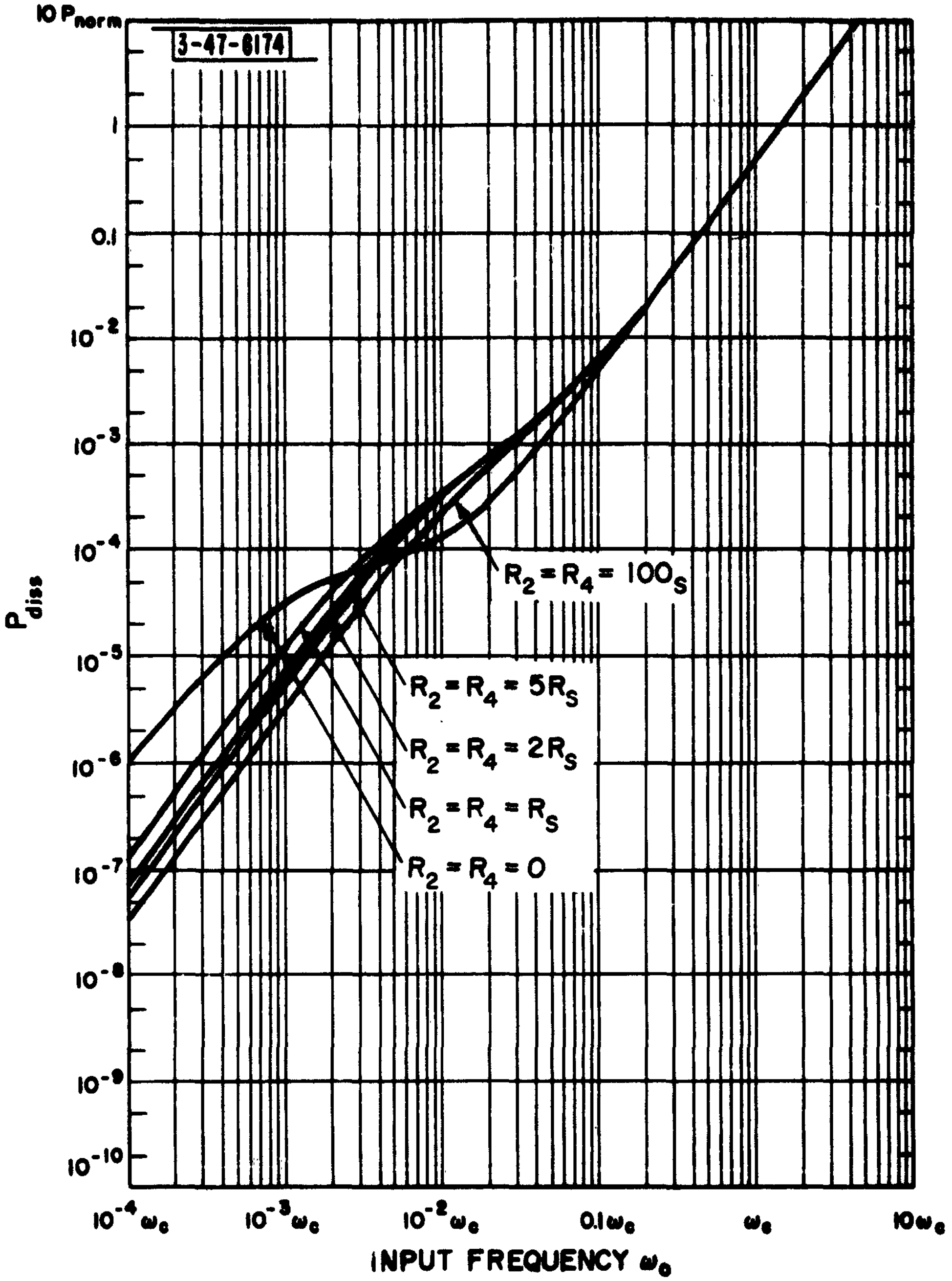

Fig. 7.6 Total power dissipated in a 1-2-4-8 octupler adjusted for maximum efficiency operation.

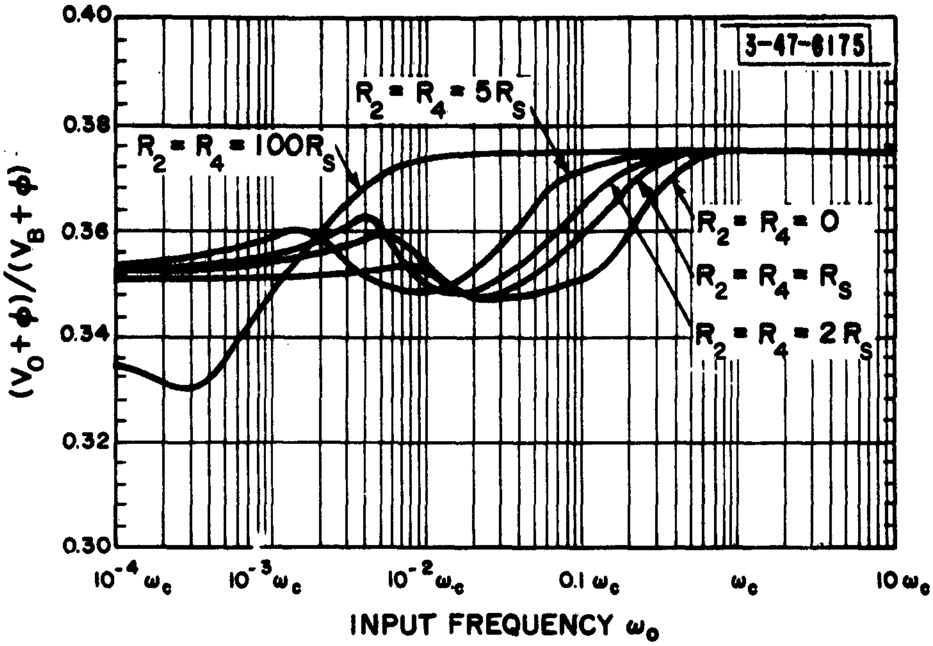

Fig. 7.7 Bias voltage for maximum efficiency operation of a 1-2-4-8 octupler for several values of the idler resistances, R_2 and R_4.

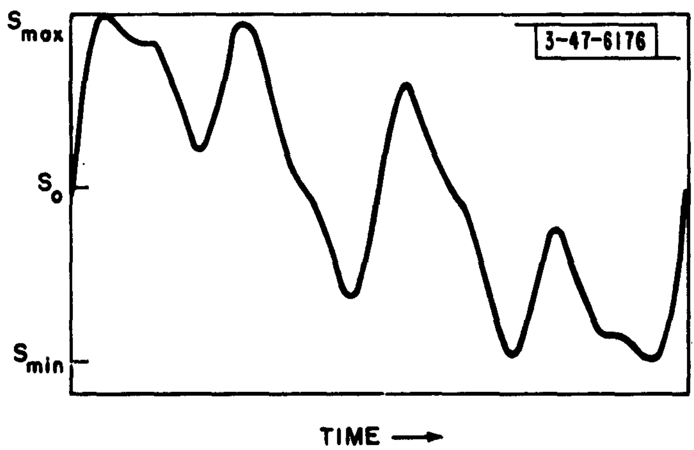

Fig. 7.8 Elastance waveform of a low-frequency 1-2-4-8 octupler adjusted for maximum efficiency operation with lossless idler terminations.

7.3 Asymptotic Formulas for the 1-2-4-8 Octupler

For low and high frequencies the performance of the octupler can be described by asymptotic formulas. These formulas are summarised in Table 7.1 for the lossless idler case ($R_2 = R_4 = 0$). At low frequencies the limiting values of the m_k (as found from the computed data) are used in Eqs. (7.4) to (7.13) to find the desired formulas. For high frequencies we use the fact that m_1 approaches 0.25 while the other m_k become small to obtain the asymptotic relations (we also use $R_{in} \approx R_8 \approx R_s$).

Table 7.1 Asymptotic Formulas for the 1-2-4-8 Octupler

Low-frequency and high-frequency formulas are given for the abrupt-junction-varactor octupler with idlers at $2\omega_o$ and $4\omega_o$. We have assumed that S_{min} is negligible, and that the idler terminations are tuned and reactive.

	Low Frequency		High Frequency
	Maximum ϵ	Maximum P_{out}	Max. ϵ and P_{out}
ϵ	$1 - 168 \frac{\omega_o}{\omega_c}$	$1 - 193 \frac{\omega_o}{\omega_c}$	$8.67 \times 10^{-19} (\frac{\omega_c}{\omega_o})^{14}$
R_{in}	$0.103 (\frac{\omega_c}{\omega_o}) R_s$	$0.116 (\frac{\omega_c}{\omega_o}) R_s$	R_s
R_8	$0.150 (\frac{\omega_c}{8\omega_o}) R_s$	$0.076 (\frac{\omega_c}{8\omega_o}) R_s$	R_s
$\frac{P_{in}}{P_{norm}}$	$0.0198 (\frac{\omega_o}{\omega_c})$	$0.0215 (\frac{\omega_o}{\omega_c})$	$0.500 (\frac{\omega_o}{\omega_c})^2$
$\frac{P_{out}}{P_{norm}}$	$0.0198 (\frac{\omega_o}{\omega_c})$	$0.0215 (\frac{\omega_o}{\omega_c})$	$4.34 \times 10^{-19} (\frac{\omega_c}{\omega_o})^{12}$
$\frac{P_{diss}}{P_{norm}}$	$3.33 (\frac{\omega_o}{\omega_c})^2$	$4.17 (\frac{\omega_o}{\omega_c})^2$	$0.500 (\frac{\omega_o}{\omega_c})^2$
$\frac{V_o + \varphi}{V_B + \varphi}$	0.351	0.352	0.375
m_1	0.155	0.153	0.250
m_2	0.103	0.116	$0.0156 \frac{\omega_c}{\omega_o}$
m_4	0.117	0.100	$3.05 \times 10^{-5} (\frac{\omega_c}{\omega_o})^3$
m_8	0.045	0.067	$2.91 \times 10^{-11} (\frac{\omega_c}{\omega_o})^7$

VIII DISCUSSION

In the preceding analyses we have made a few assumptions so that we could derive fundamental performance limits for varactor frequency multipliers. Some of these restrictions can be removed to give better estimates of the performance which can be obtained in practical devices. For example, coupling network losses were neglected in the theory except at the idler frequencies. These losses can be accounted for as follows: 1. Lump the input circuit losses with the "input resistance" to obtain the actual input resistance; 2. Compute the actual load resistance as the difference between the theoretical load resistance and the output circuit losses; 3. Determine input and output circuit efficiencies from the known values of the losses, the load resistance, and the "input resistance"; 4. Calculate the expected overall efficiency as the product of the theoretical efficiency with lossy idler circuits, the input circuit efficiency, and the output circuit efficiency.

Case capacitance presents a more difficult problem. For low-frequency varactors the case capcitance is usually small compared to the nonlinear capacitance, and therefore its effect is small. However, high-frequency varactors usually have case capacitances of approximately the same values as the nonlinear capacitances, and we therefore expect the case capacitance to have a significant effect on performance. The best way to handle this problem, from a theoretical standpoint, is to tune out the case capacitance at all frequencies of importance. This approach, however, is not always feasible in practice, particularly when the coupling networks are realized with distributed parameter elements (coaxial line or waveguide, for example). If the case capacitance is not negligible and it is not tuned out, then its degrading effect on performance can be determined by the usual techniques of circuit theory.

Lead inductance is a very important element when fundamental limits on bandwidth are sought. However, for narrow-band applications, it is relatively unimportant because it can be simply included in the series tuned circuit at each frequency.

Throughout the analysis tuned conditions were assumed in the output and idler circuits. This assumption is not necessarily valid, since more

favorable current amplitudes may be obtained in a detuned mode of operation. In fact, there may be significant advantages to be gained by not tuning the idler circuits (bandwidth, for example, may be improved). The addition of the phase variables in the untuned condition makes the theory much more complicated. Furthermore, at low frequencies the efficiency is already so high that any improvement would be negligible. At high frequencies all harmonic currents, except the fundamental, become small so that detuning would have little or no effect. In the mid-frequency range some improvement in efficiency might be expected (perhaps 2 or 3 per cent).

In analyzing the various multipliers, we have tacitly assumed an idealized coupling network of the general type depicted in Fig. 1.2. Obviously, in practice network realizations will be quite different. Idler circuits will very often be built into the input and output circuits. Also, there will very frequently be impedance transformations between the varactor terminals and the load and source terminals. In any case, however, the coupling networks and the actual load and source impedances should be such that the constraints of the theory are satisfied, that is, the impedances seen at the varactor terminals should be the optimum values as given by the theory regardless of the other details of the coupling network.

The preceding theoretical treatment assumed that the varactor was fully driven. If the available drive power is not sufficient for full drive, then the theory must be modified. For a particular varactor this can be done by considering an equivalent, fully-driven varactor with different values of S_{min}, S_{max}, V_B, and V_{min} (same R_s). Obviously, the cutoff frequency (or figure of merit), ω_c, will be reduced as will the normalization power. The equations, however, will remain the same and the graphs will still be applicable when properly modified by the new values of S_{min}, S_{max}, V_B, V_{min}, ω_c, and P_{norm}. The actual calculation of the performance may be quite difficult, since the new values of ω_c and P_{norm} must be chosen to be compatible with the specified drive level. A trial and error procedure will probably be required, but convergence to the desired result should be rapid.

We have disregarded one important mode of operation of the varactor which is not yet fully understood. This is the overdriven case when the conduction mechanism of the varactor fails. When this occurs (usually at quite high frequencies), the power input can, in many instances, be considerably increased with little or no deterioration of performance (sometimes there is an improvement). This is believed to be the result of charge storage, i. e., the input power is not converted to d. c. power by rectification but is stored in the varactor as useful energy which can be returned as harmonic energy when the current direction reverses. A particularly advantageous feature of this type of operation is that performance appears to be relatively insensitive to power level changes which is not the case with normal operation in the non-conducting region. As this phenomenon becomes better understood it will probably become a more important factor in the design of practical devices. (A more detailed discussion of the implications of this phenomenon in practical devices has been given by Penfield.[10])

8.1 Comparison of the Efficiencies of the Various Multipliers

It is interesting to compare the various abrupt-junction-varactor multipliers we have described. For generality we also compare their performance with those of the abrupt-junction-varactor doubler[5] and the graded-junction-varactor doubler.[3]

In Fig. 8.1 we plot the maximum efficiency of the abrupt-junction-varactor doubler, tripler, 1 2-4 quadrupler, 1-2-3-4 quadrupler, 1-2-4-5 quintupler, 1-2-4-6 sextupler, 1-2-3-6 sextupler, and 1-2-4-8 octupler. We also show the maximum efficiency of the graded-junction-varactor doubler. These are plotted as functions of output frequency rather than input frequency. Lossless idler terminations are assumed in each case.

At low frequencies all of these curves have the asymptotic behavior,

$$\epsilon \approx 1 - \alpha \frac{\omega_{out}}{\omega_c} \tag{8.1}$$

where α is not a constant for all mutlipliers but depends on the particular type and order of the multiplier. The values of α for the various multipliers

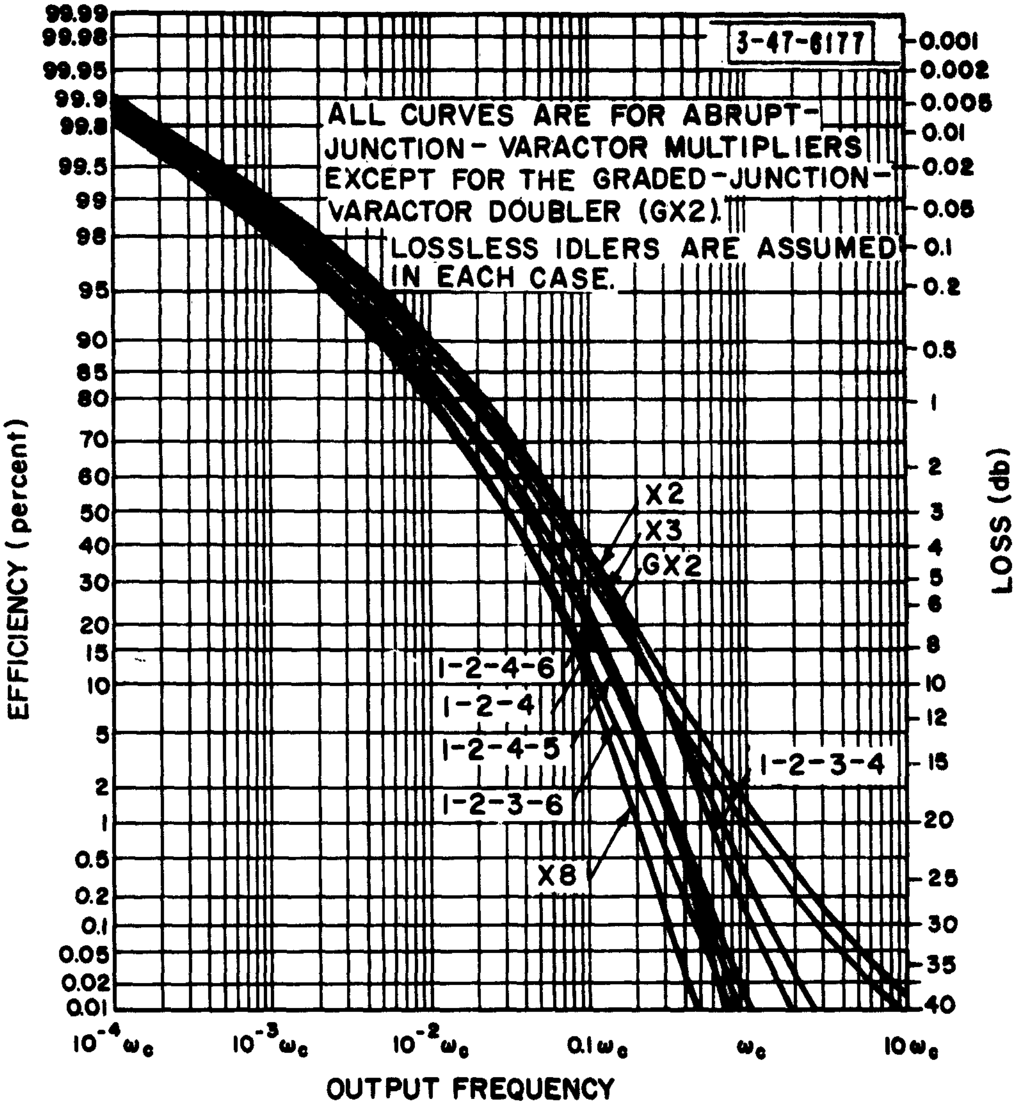

Fig. 8.1 Maximum efficiency for several abrupt-junction-varactor multipliers. Also shown is the efficiency of the graded-junction-varactor doubler. All idler terminations are assumed to be lossless.

are given in Table 8.1. (The cascade figure of merit, $\alpha n/(n-1)$, given in Table 8.1 will be discussed shortly.)

Table 8.1 Cascade Figures of Merit

In this table we give values of α and $\alpha n/(n-1)$ for the various abrupt-junction-varactor multipliers which have been solved and for the graded-junction-varactor doubler. Lossless idler terminations have been assumed.

Multiplier	α	$\frac{n}{n-1}\alpha$
X2 (Graded)	13.0	26.0
X2	9.95	19.9
X3	11.6	17.4
X4 (1-2-4)	15.6	20.8
X4 (1-2-3-4)	11.4	15.2
X5 (1-2-4-5)	18.6	23.3
X6 (1-2-4-6)	16.6	19.9
X6 (1-2-3-6)	22.3	26.8
X8 (1-2-4-8)	21.0	24.0

An alternate formula for the low-frequency efficiencies can be found by observing that the right-hand side of Eq. (8.1) is simply the first two terms of the power series expansion of $e^{-\alpha(\omega_{out}/\omega_c)}$. Thus, we try

$$\epsilon = e^{-\alpha(\omega_{out}/\omega_c)} \tag{8.2}$$

as a possible low-frequency asmptotic relation for the multipliers.[8] Empirically, it is found that Eq. (8.2) has a wider range of validity than Eq. (8.1); and, in fact, Eq. (8.2) agrees within approximately 3 per cent with the efficiencies given in Fig. 8.1 for output frequencies up to about $0.1\,\omega_c$.

The overall efficiency of a complete multiplier chain is the product of the efficiencies of the individual stages. Since the exponents add when exponentials are multiplied together, Eq. (8. 2) is an especially convenient form for calculating the overall efficiency. For the special case of a cascade of $\underline{m}$ multipliers each of order $\underline{n}$, we can write[8]

$$\epsilon = e^{-(n + n^2 + \ldots + n^m)\,\alpha(\omega_o/\omega_c)} , \tag{8. 3}$$

where we have assumed that all varactors are optimally driven and that they all have the same cutoff frequency. The sum in the exponent can be expressed in closed form, i. e. , we have

$$\epsilon = e^{-\alpha \frac{n}{n-1} \frac{\omega_o}{\omega_c} (n^m - 1)} , \tag{8. 4}$$

or,

$$\epsilon = e^{-\alpha \frac{n}{n-1} \frac{\omega_{out} - \omega_o}{\omega_c}} . \tag{8. 5}$$

We see that the quantity $\alpha n/(n-1)$ is a measure for comparing cascades of various types of multipliers operating between the same two frequencies. The values of $\alpha n/(n-1)$ are summarized in Table 8. 1.[9] The near equality of the values of $\alpha n/(n-1)$ shows that the theoretical efficiency of a chain of multipliers does not depend much upon order of multiplication for low-frequency operation. Thus, the choice of order of multiplication will not be made on the basis of theoretical efficiencies, at least for frequencies low enough for Eq. (8. 5) to be applicable.

For a cascade of two different multipliers we can write,

$$\begin{aligned} \epsilon &= e^{-(\alpha_1 n_1 \omega_o/\omega_c \; + \; \alpha_2 n_1 n_2 \omega_o/\omega_c)} \\ &= e^{-(\omega_{out}/\omega_c)(\alpha_1/n_2 + \alpha_2)} , \end{aligned} \tag{8. 6}$$

where n_1 is the order of multiplication of the first multiplier which is characterized by the constant α_1 (similarly for n_2 and α_2). For a cascade of three different multipliers we have,

$$\epsilon = e^{-\frac{\omega_{out}}{\omega_c}\left(\frac{\alpha_1}{n_2 n_3} + \frac{\alpha_2}{n_3} + \alpha_3\right)} \quad . \tag{8.7}$$

Equations (8.6) and (8.7) can be generalized to the form,

$$\epsilon = e^{-\alpha_c \omega_{out}/\omega_c} \quad , \tag{8.8}$$

where the cascade parameter, α_c, is given by the formula,

$$\alpha_c = \frac{\alpha_1}{n_2 n_3 \ldots n_k} + \frac{\alpha_2}{n_3 n_4 \ldots n_k} + \ldots + \frac{\alpha_{k-1}}{n_k} + \alpha_k \quad . \tag{8.9}$$

We can now use Eq. (8.8) to compare various cascades of multipliers operating between the same two frequencies. For example, we may compare the performance of three cascaded doublers or a cascade of a doubler and a quadrupler with the performance of the octupler. There are various ways to combine the doubler with one or the other of the two quadruplers (doubler followed by a 1-2-4 quadrupler, 1-2-3-4 quadrupler followed by a doubler, etc.). The cascade parameters, α_c, for the various possible ways of multiplication by four, six, and eight are given in Table 8.2.

We see with the aid of Table 8.2 and Eq. (8.8) that the efficiencies of the various schemes for generating the fourth, sixth, or eighth harmonic are about the same. The choice between a cascade of multipliers and a single higher-order multiplier must therefore be based on practical considerations. At low frequencies where the above formulas apply, the use of many stages has the advantage that each stage is easier to design and that less power is dissipated in each varactor. On the other hand, coupling network losses will probably more than offset the slight advantage in efficiency of the cascaded

Table 8.2 Cascade Parameters for Multiplication by Four, Six, and Eight

In this table we give values for the cascade parameter, α_c, as defined by Eq. (8.9) for the various possible schemes of generating the fourth, sixth, and eighth harmonics. Lossless idler terminations have been assumed and the varactors are all assumed to have the same cutoff frequency.

Sequence of multipliers in the cascade	Cascade parameter, α_c
Multiplication by four	
X2-X2	14.9
X4 (1-2-4)	15.6
X4 (1-2-3-4)	11.4
Multiplication by six	
X2-X3	14.9
X3-X2	15.7
X6 (1-2-4-6)	16.6
X6 (1-2-3-6)	22.3
Multiplication by eight	
X2-X2-X2	17.4
X2-X4 (1-2-4)	18.1
X2-X4 (1-2-3-4)	13.9
X4 (1-2-4)-X2	17.8
X4 (1-2-3-4)-X2	15.6
X8 (1-2-4-8)	21.0

multipliers. Also, symmetrical circuits can be used to split the power dissipation among two (or more) varactors in a higher-order multiplier.[11]

It is readily apparent that we do not have this flexibility of choice when the two frequencies involved are related by a prime number. The desired harmonic can still be generated directly with a single higher-order multiplier with idler circuits. However, this approach is not very feasible in practice when the integer involved is large because of the difficulty of controlling more than two or three idler currents. This difficulty can be circumvented to a large degree by using large-signal upconverters to sum the outputs of two multipliers (or multiplier chains). In this way we can generate high-order harmonics which are prime numbers with simpler lower-order multipliers and a large-signal upconverter.[12]

8.2 Comparison of the Power Outputs of the Various Multipliers

The multipliers can also be compared on the basis of power output. Figure 8.2 shows the power outputs of the various multipliers for maximum efficiency operation with lossless idler terminations. The curves are plotted versus <u>input</u> frequency. At low frequencies the lower order multipliers are seen to have slightly greater power outputs.

At low frequencies the power outputs of all the multipliers vary linearly with frequency, each with a different constant of proportionality. The appropriate low-frequency asymptotic formulas are given in Table 8.3. In this table we have expressed the powers both in terms of P_{norm} and ω_c and in terms of Uhlir's nominal reactive power, P_r:[8]

$$P_r = \frac{(V_B + \varphi)^2}{2S_{max}} = \frac{S_{max}}{2(S_{max}^2 - S_{min}^2)} (V_B - V_{min})^2$$

$$= \frac{1}{2} \frac{S_{max}}{S_{max} + S_{min}} \cdot \frac{P_{norm}}{\omega_c} \xrightarrow[S_{min} \to 0]{} \frac{1}{2} \frac{P_{norm}}{\omega_c} . \tag{8.10}$$

Note that P_r has the dimensions of power per unit frequency, and must therefore be multiplied by a frequency to yield power.

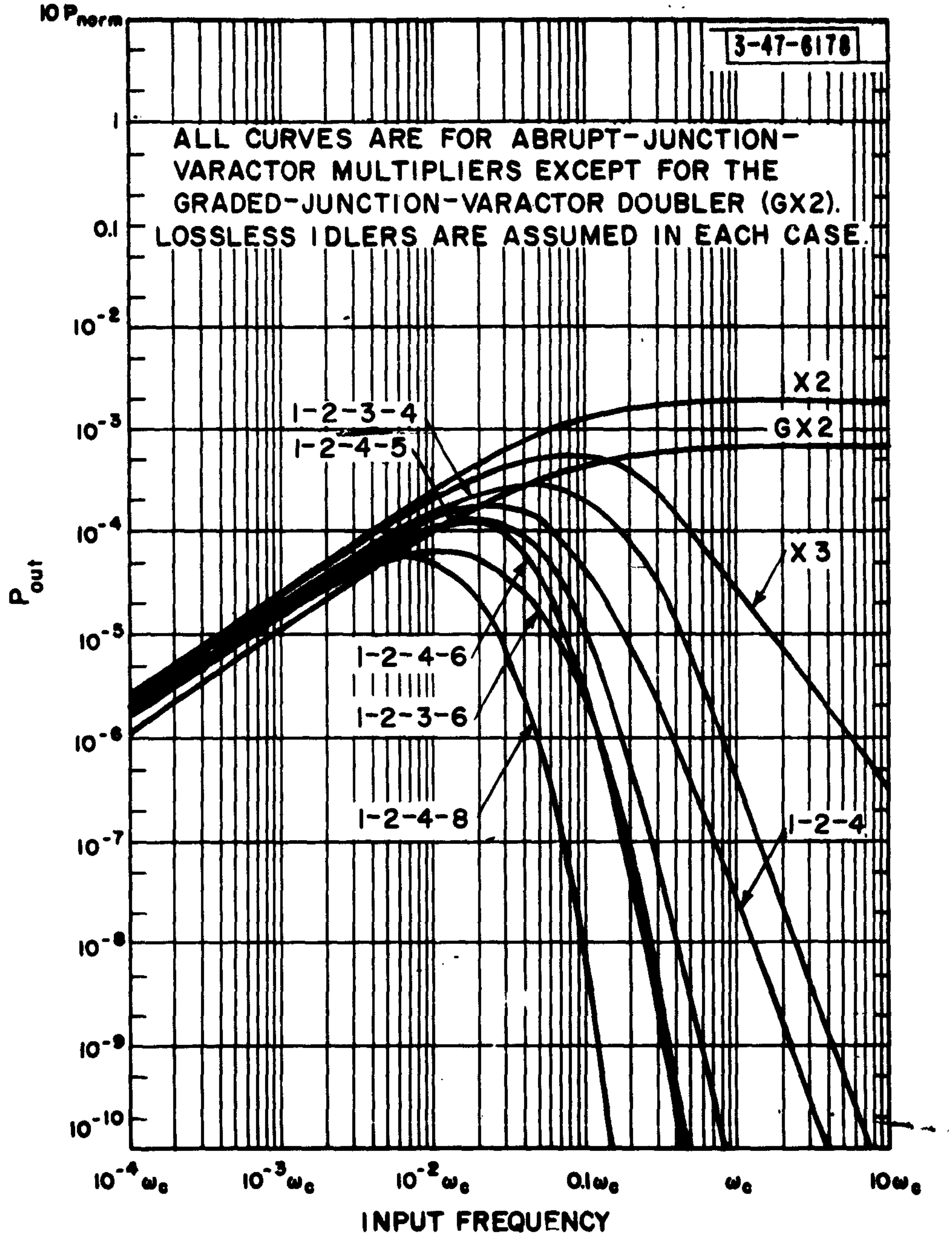

Fig. 8.2 Output power for several abrupt-junction-varactor multipliers and the graded-junction-varactor doubler as a function of the input frequency. All are assumed to operate at maximum efficiency with lossless, tuned idler terminations.

Table 8.3 Low-Frequency Power Output Formulas

Low-frequency power output formulas are given for the various multipliers. The formulas are for maximum efficiency operation with lossless idler terminations. We have assumed that $S_{min} << S_{max}$.

Multiplier	Low-Frequency Power Output
X2 (Graded)	$0.012\, P_{norm} \frac{\omega_o}{\omega_c} = 0.024\, P_r \omega_o = 0.15\, P_r f_o$
X2	$0.028\, P_{norm} \frac{\omega_o}{\omega_c} = 0.057\, P_r \omega_o = 0.36\, P_r f_o$
X_3	$0.024\, P_{norm} \frac{\omega_o}{\omega_c} = 0.048\, P_r \omega_o = 0.30\, P_r f_o$
X4 (1-2-4)	$0.020\, P_{norm} \frac{\omega_o}{\omega_c} = 0.040\, P_r \omega_o = 0.25\, P_r f_o$
X4 (1-2-3-4)	$0.0226\, P_{norm} \frac{\omega_o}{\omega_c} = 0.045\, P_r \omega_o = 0.28\, P_r f_o$
X5 (1-2-4-5)	$0.018\, P_{norm} \frac{\omega_o}{\omega_c} = 0.036\, P_r \omega_o = 0.22\, P_r f_o$
X6 (1-2-4-6)	$0.022\, P_{norm} \frac{\omega_o}{\omega_c} = 0.045\, P_r \omega_o = 0.28\, P_r f_o$
X6 (1-2-3-6)	$0.018\, P_{norm} \frac{\omega_o}{\omega_c} = 0.037\, P_r \omega_o = 0.23\, P_r f_o$
X8 (1-2-4-8)	$0.021\, P_{norm} \frac{\omega_o}{\omega_c} = 0.043\, P_r \omega_o = 0.27\, P_r f_o$

8.3 High-Frequency Limit

The high-frequency limit is inherently less interesting than the low-frequency limit because the efficiency is very small and the power dissipation is very large. Nevertheless, in some applications these limits are pertinent.

At high frequencies the series resistance dominates, and to transfer maximum power to the load or to obtain maximum efficiency, the load resistance should equal R_s. The input resistance is also equal to R_s at high frequencies.

In deriving high-frequency asymptotic relations for the various abrupt-junction-varactor multipliers, we used the facts that $m_1 \approx 0.25$, $R_{in} \approx R_{load} \approx R_s$, and $m_k << m_1$ for $k > 1$. We then found that the m_k (for $k > 1$) are proportional to $(\omega_c/\omega_o)^{k-1}$ and, therefore, approach zero. By the same techniques it can be shown that the m_k fall according to this relation for any value of k. This allows us to write the power output into a matched load as

$$P_{out} = 8\ell^2 m_\ell^2 P'_{norm} \left(\frac{\omega_o}{\omega_c}\right)^2 \tag{8.11}$$

or

$$P_{out} \sim P'_{norm} \left(\frac{\omega_c}{\omega_o}\right)^{2(\ell-2)} . \tag{8.12}$$

The doubler ($\ell = 2$) has a power output which approaches a finite limit, while P_{out} for higher-order multipliers goes to zero as $(\omega_c/\omega_o)^{2(\ell-2)}$. At high frequencies the power input is the same for all multipliers:

$$P_{diss} \approx P_{in} \approx 0.5\, P'_{norm} \left(\frac{\omega_o}{\omega_c}\right)^2 \tag{8.13}$$

The efficiency therefore becomes

$$\epsilon = 16\ell^2 m_\ell^2 \tag{8.14}$$

$$\sim \left(\frac{\omega_c}{\omega_o}\right)^{2(\ell-1)} . \tag{8.15}$$

It is apparent that multipliers with idlers have efficiencies that approach zero very rapidly at high frequencies. These efficiencies can be improved by cascading multipliers or by using varactors with much sharper nonlinearities to convert power directly into the desired harmonic. In the first case, for example, the efficiency of a quadrupler falls as $(\omega_c/\omega_o)^6$, while that of two cascaded doublers falls as $(\omega_c/\omega_o)^4$.

One important result of the high-frequency limit is that it clearly shows the superiority of the doubler as a high-frequency multiplier (see Fig. 8.1 and the tables of high-frequency asymptotic formulas in the various chapters). Thus, the last stage (or, perhaps, the last 2 or 3 stages) should be constructed with doublers if a high output frequency is involved. At high frequencies the doubler efficiency falls as $0.0039(\omega_c/\omega_o)^2$ and the 1-2-4 quadrupler efficiency decreases as $5.96 \times 10^{-8}(\omega_c/\omega_o)^6$. Two cascaded doublers thus give an efficiency of $3.8 \times 10^{-6}(\omega_c/\omega_o)^4$. For operating frequencies near ω_c the two doublers in cascade are approximately two orders of magnitude better than the quadrupler. For lower frequencies the difference is not so pronounced, but it still exists. Figure 8.1 indicates that doublers are definitely to be preferred when the <u>output</u> frequency is in the vicinity of (or greater than) two- or three-tenths of the cutoff frequency.

8.4 Summary

In the preceding analyses we have derived several important results, the most important being the fact that varactors are capable of yielding high efficiencies at lower frequencies when used as frequency multipliers. We have derived formulas and found numerical solutions of the pertinent equations for several of the multiplier configurations which are used in practice. In the process of analyzing the various multipliers we have found that the 1-2-3-4 quadrupler is definitely superior to the 1-2-4 quadrupler and that the 1-2-4-6 sextupler is better than the 1-2-3-6 sextupler at low frequencies.

Furthermore, it has been shown that the 1-2-3-5 quintupler has an anomaly which probably prevents it from being a practical low-frequency multiplier. An investigation of low-frequency efficiency formulas has revealed that cascaded multipliers and single higher-order multipliers are nearly equivalent on the basis of efficiency.

There are several aspects of multipliers, such as bandwidth, noise, and operation into the forward region of the varactor upon failure of the rectification mechanism, which we have acknowledged as distinct problems but have not pursued. They are, in fact, very difficult problems and each will need considerable study before any definite conclusions concerning fundamental limits can be reached.

An important practical problem which we have not discussed is that of the circuit design of multipliers. There are obviously many varieties of coupling networks which can be visualized each of which will satisfy the current and impedance constraints set by the varactor. We have derived the necessary current and impedance constraints, but we have not pursued the network realization problem beyond this point.

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